COVARIANT LYAPUNOV VECTORS AS A MEANS TO CHARACTERIZE SPACE-TIME CHAOS

Antonio Politi
CNR, Istituto dei Sistemi Complessi, Firenze

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Coll.: H. Chaté, F. Ginelli, R. Livi, P. Poggi, A. Torcini, A. Turchi, R. Zillmer

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- LYAPUNOV EXPONENTS AS ENSEMBLE AVERAGES
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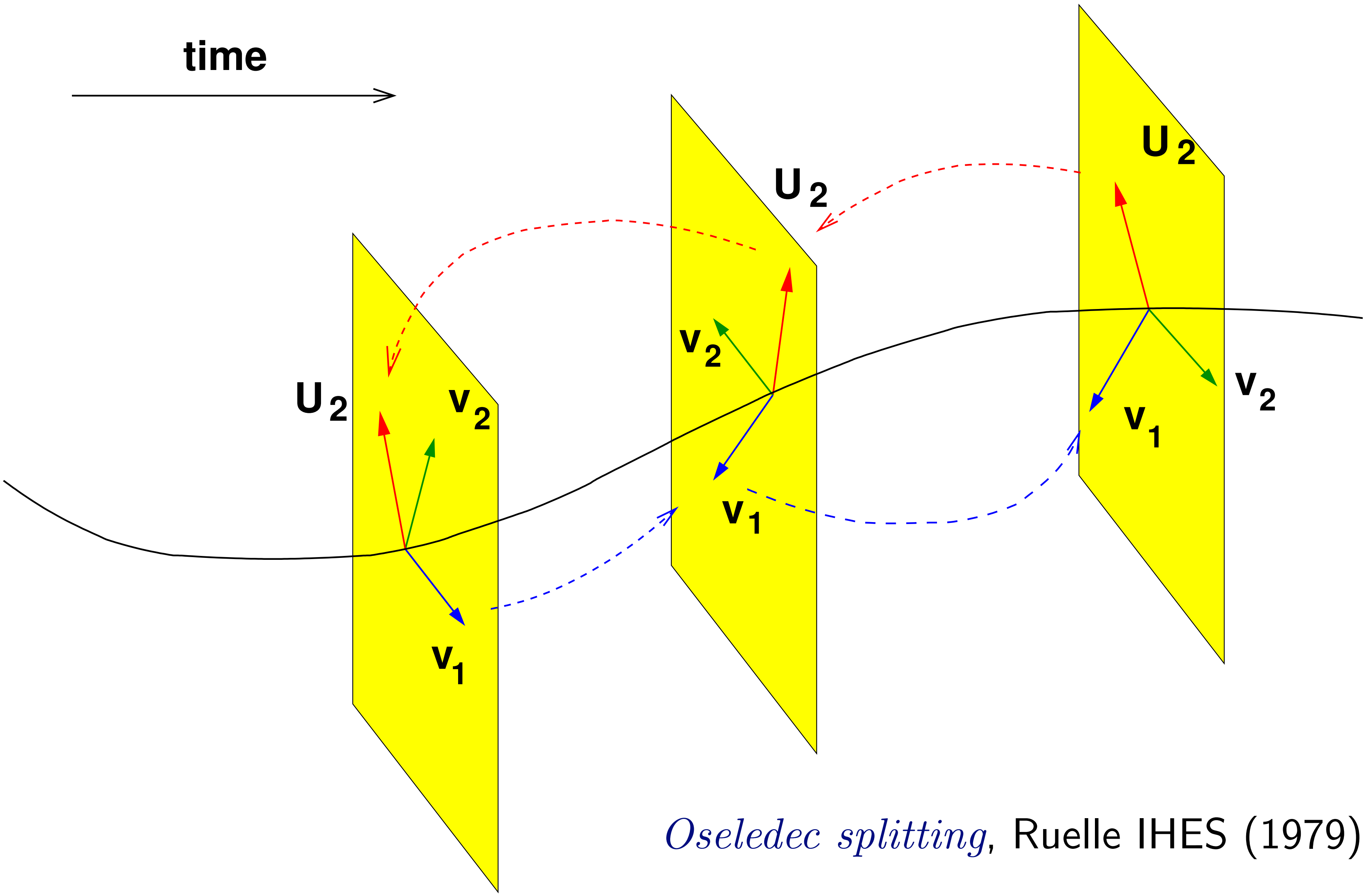
- LOCALIZATION OF COVARIANT LYAPUNOV VECTORS
- A WAY TO IDENTIFY COLLECTIVE DEGREES OF FREEDOM?

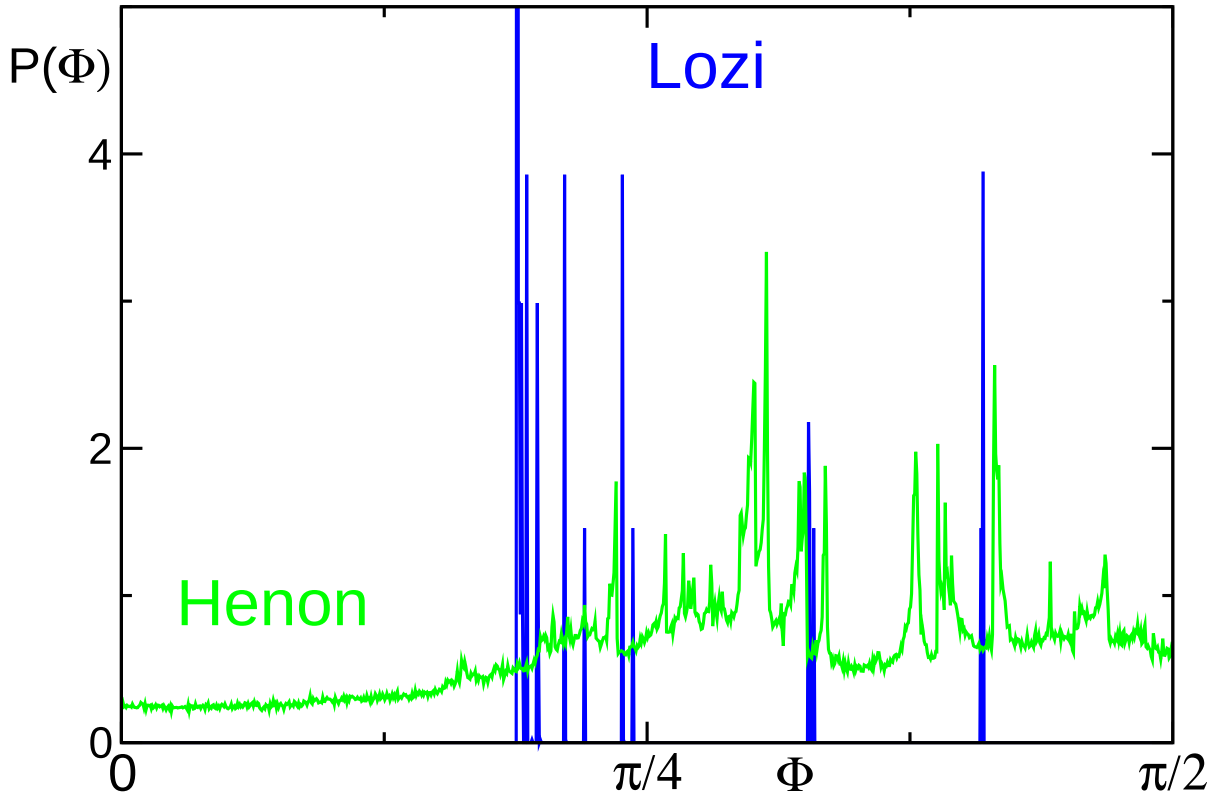
THE METHOD
(nothing new under the sun)



Oseledec splitting, Ruelle IHES (1979)

TRANSEVERSALITY BETWEEN STABLE AND UNSTABLE MANIFOLD

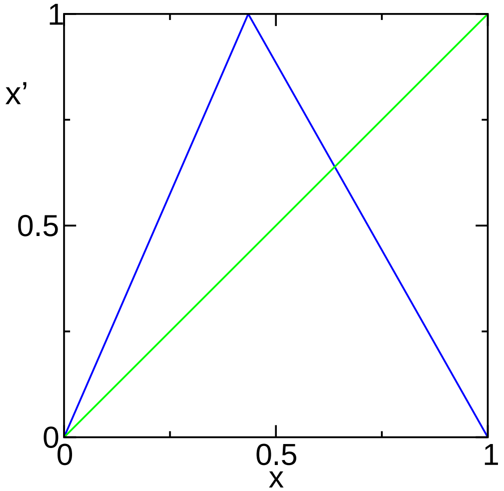
$$\phi_n^{j,k} = |\mathrm{acos}|\mathbf{u}_n^j \mathbf{u}_n^k||$$



SPATIALLY EXTENDED SYSTEMS: DISCRETE-TIME MODELS

$$x^i_{n+1} = (1-2\varepsilon)f(x^i_n) + \varepsilon[f(x^{i-1}_n) + f(x^{i+1}_n)]$$

tent maps



$$p^i_{n+1} = p^i_n + \mu[g(q^{i+1}_n - q^i_n) + g(q^i_n - q^{i-1}_n)]$$

$$q^i_{n+1} = q^i_n + p^i_{n+1}$$

symplectic maps $\qquad g(z) = \frac{\sin(2\pi z)}{2\pi}$

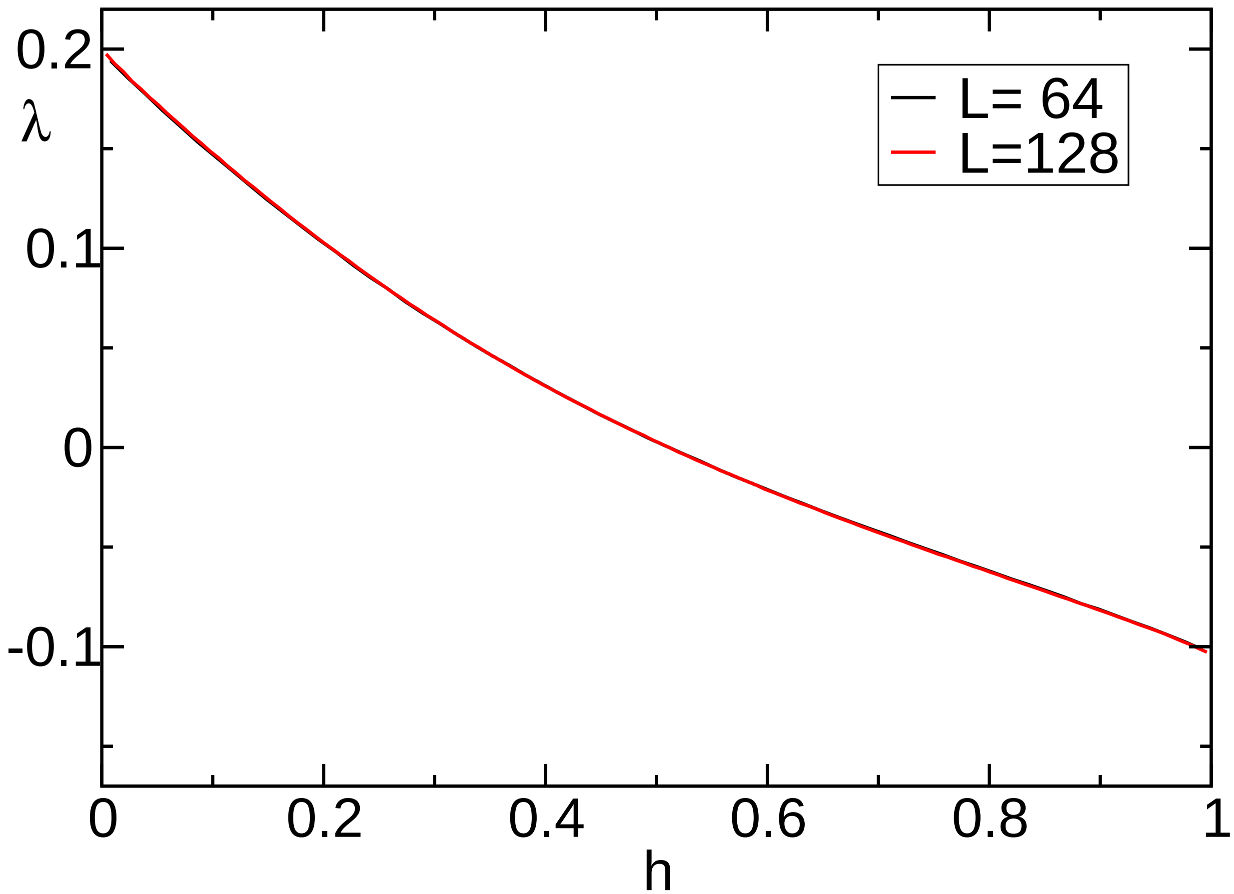
SPATIALLY EXTENDED SYSTEMS: CONTINUOUS-TIME MODELS

$$\ddot{q}_i = F(q_{i+1} - q_i) - F(q_i - q_{i-1})$$

$F(x) = \sin x$ coupled rotators

$F(x) = x + x^3$ FPU-β model

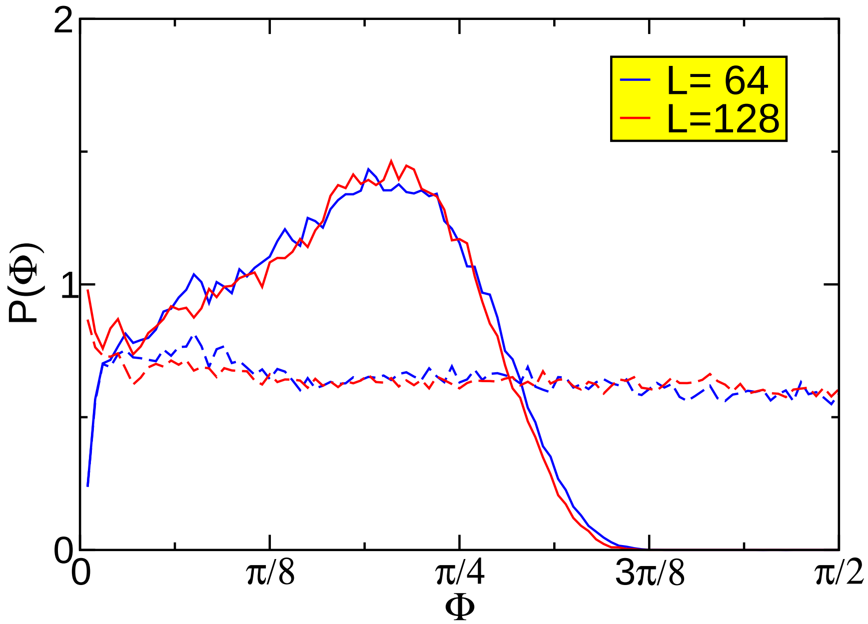
Tent maps - Lyapunov spectrum

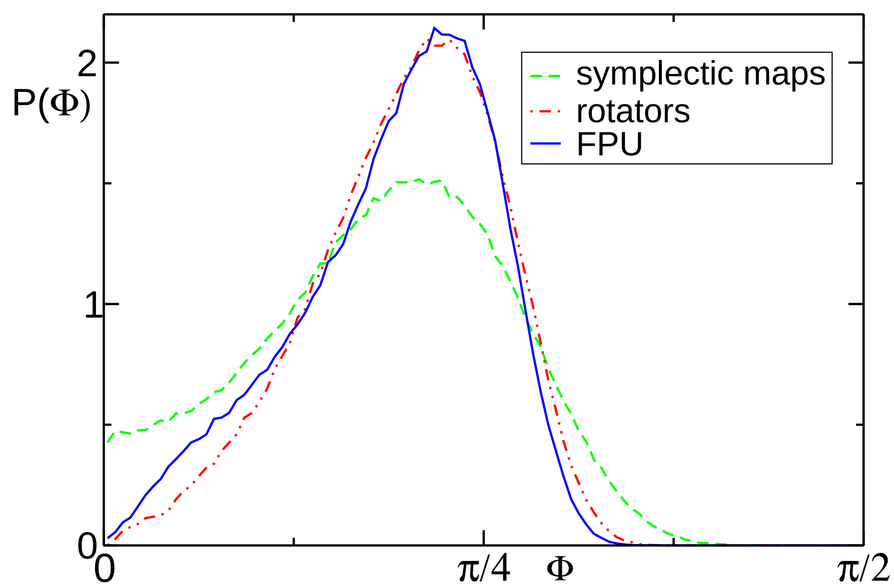


TRANSVERSALITY CHECK

$$\Phi = \min\left\{\phi_n^{j,k}, |\lambda_j > 0 \quad , \lambda_k < 0\right\}$$

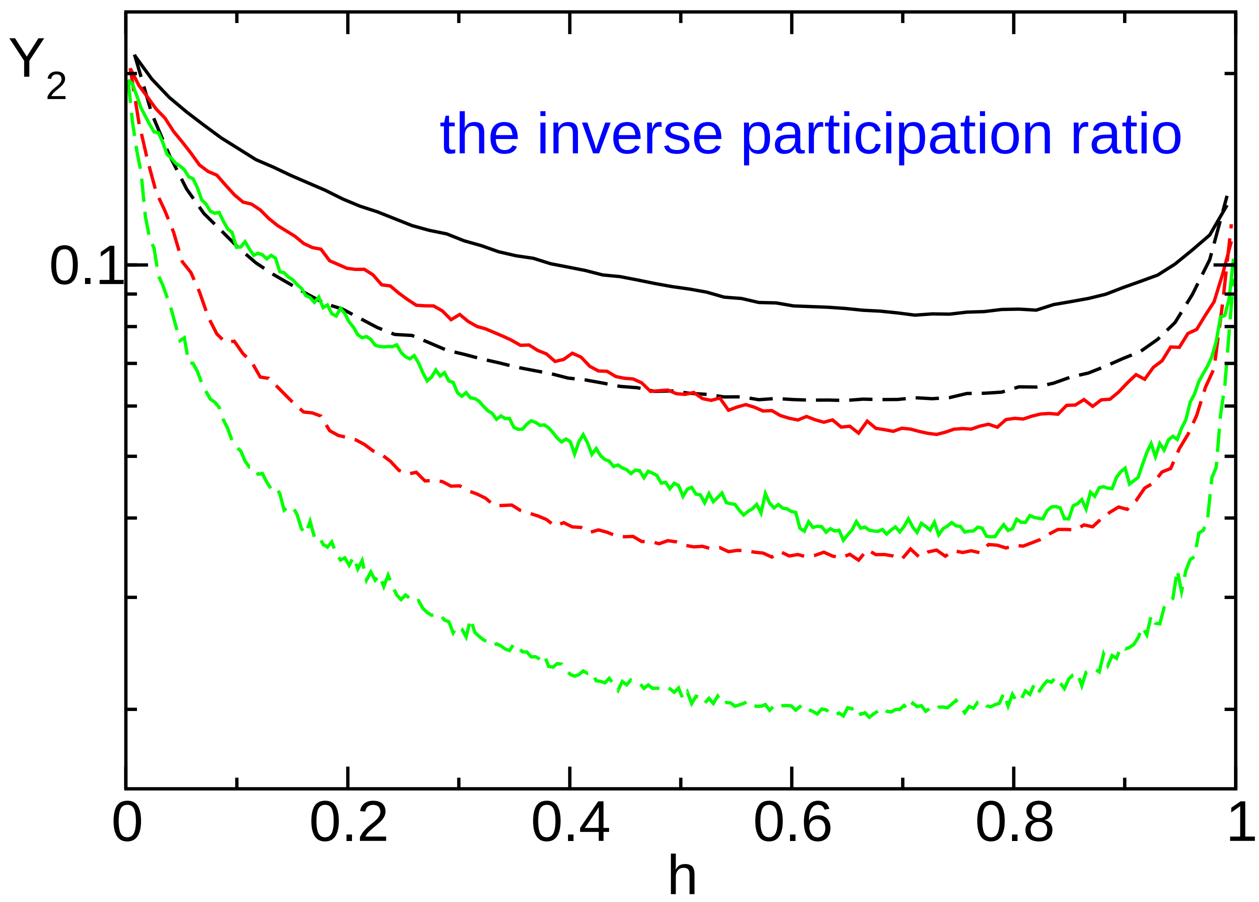
tent maps



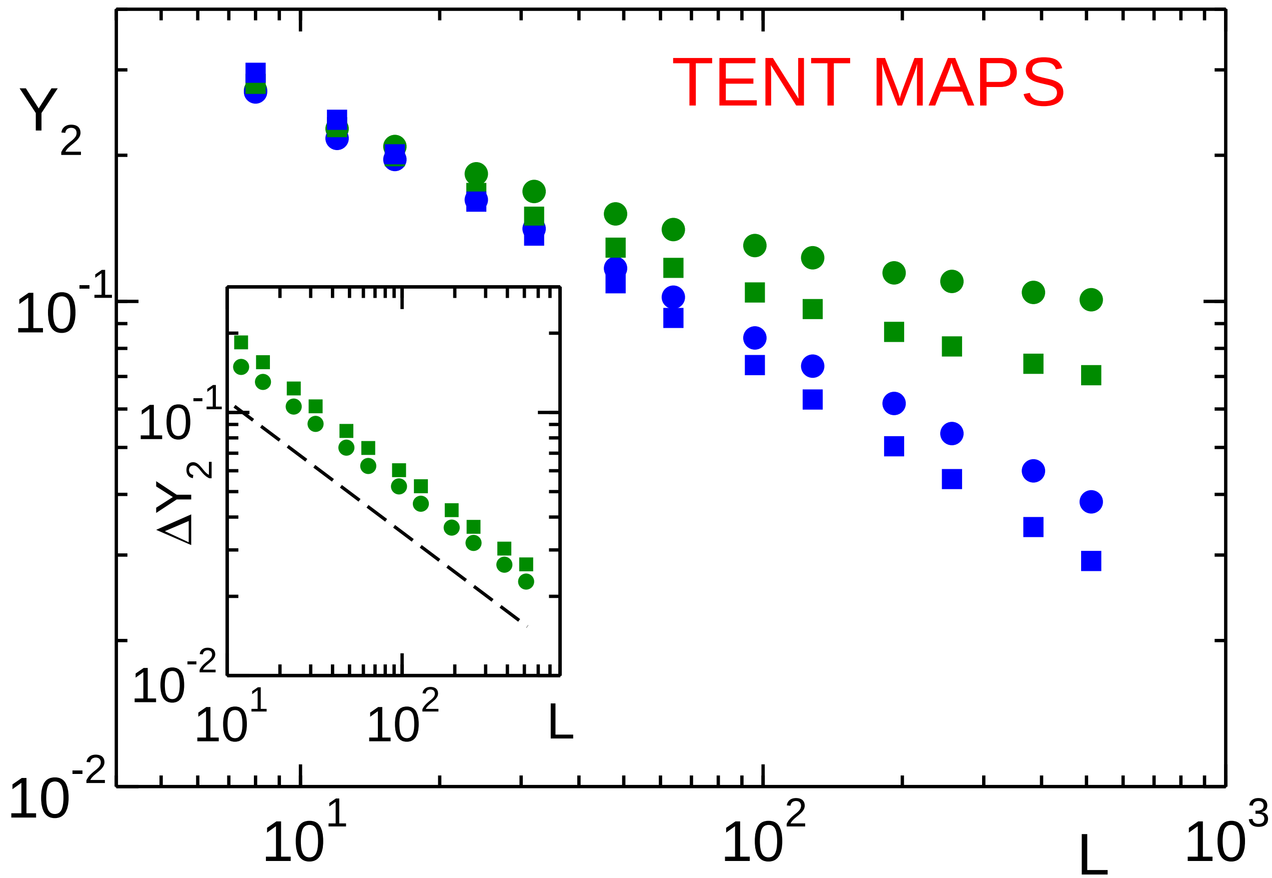


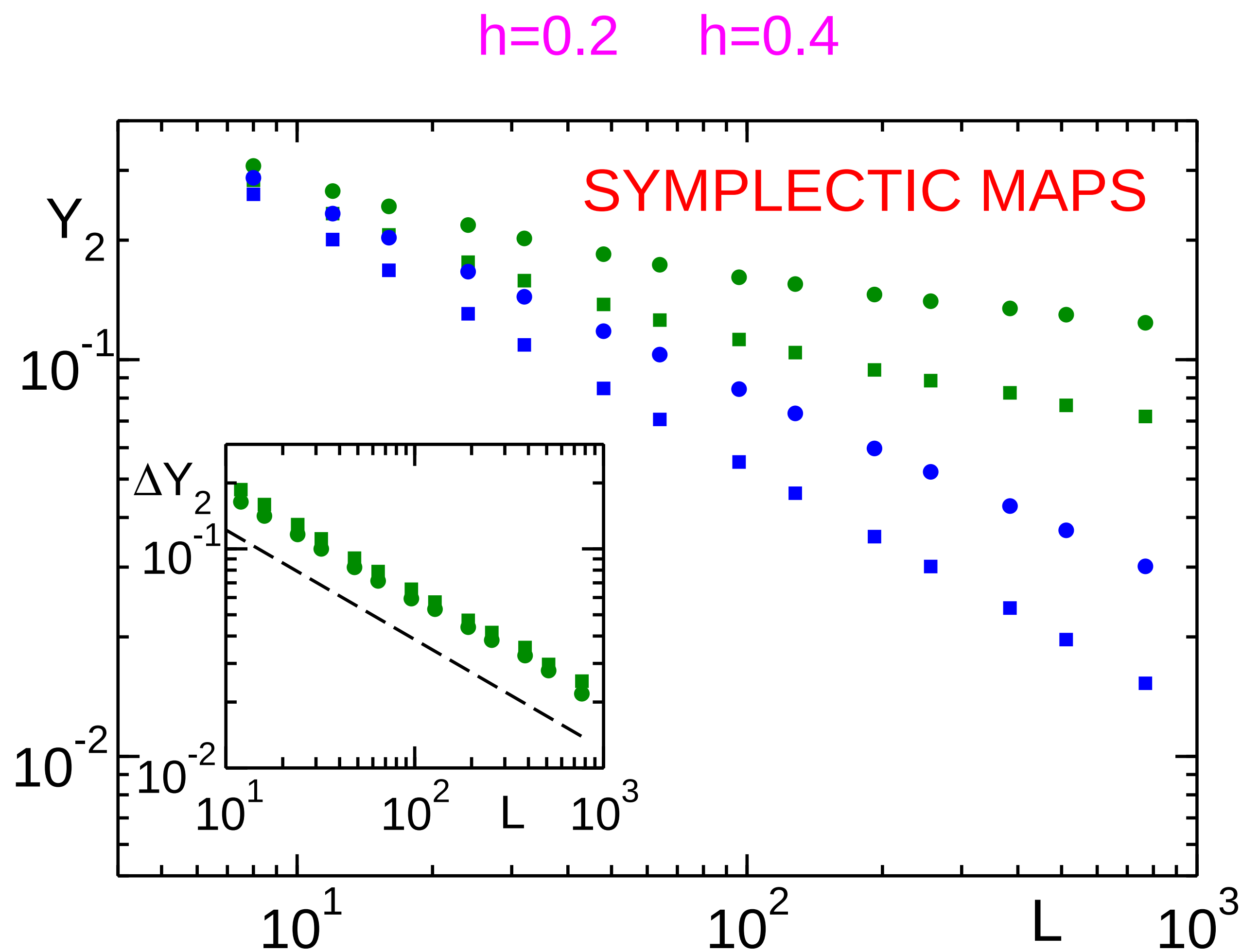
LOCALIZATION OF THE COVARIANT VECTORS

$$Y_2 = \sum_i |u_i^j|^4 \qquad\qquad \sum_i |u_i^j|^2 = 1$$

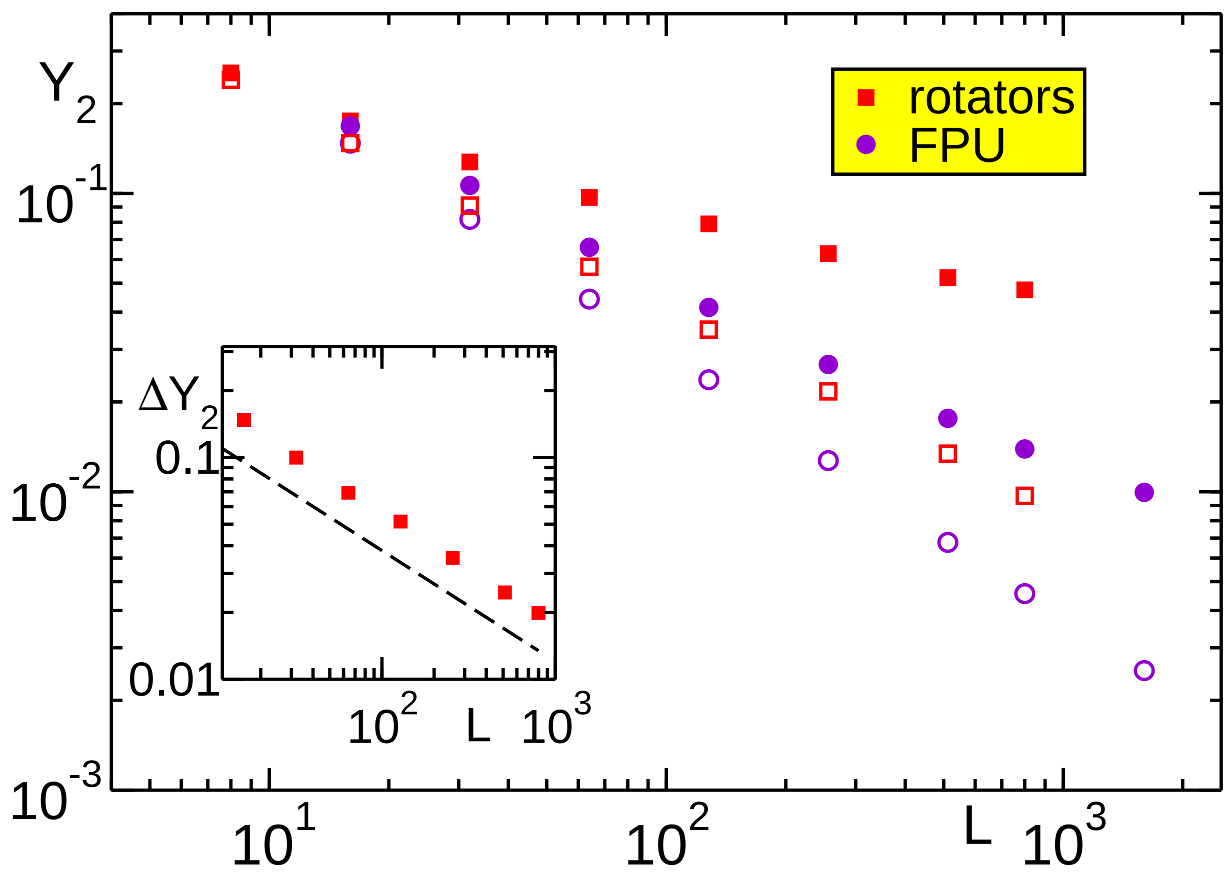


h=0.2 h=0.4

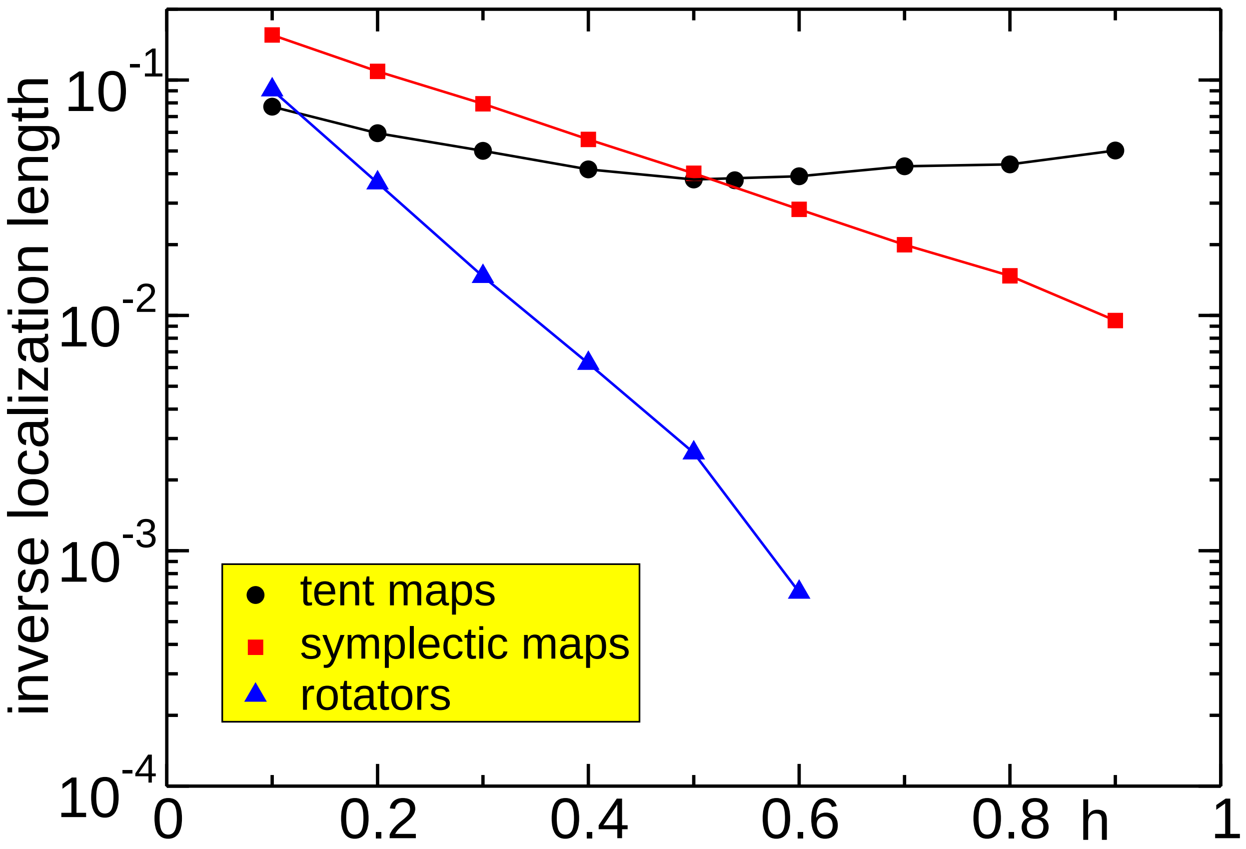



h=0.2 h=0.4
SYMPLECTIC MAPS
Y_2
10^{-1}
10^{-2}
10^1
10^2
L
10^3
ΔY_2
10^{-1}
10^{-2}
10^1
10^2
L
10^3

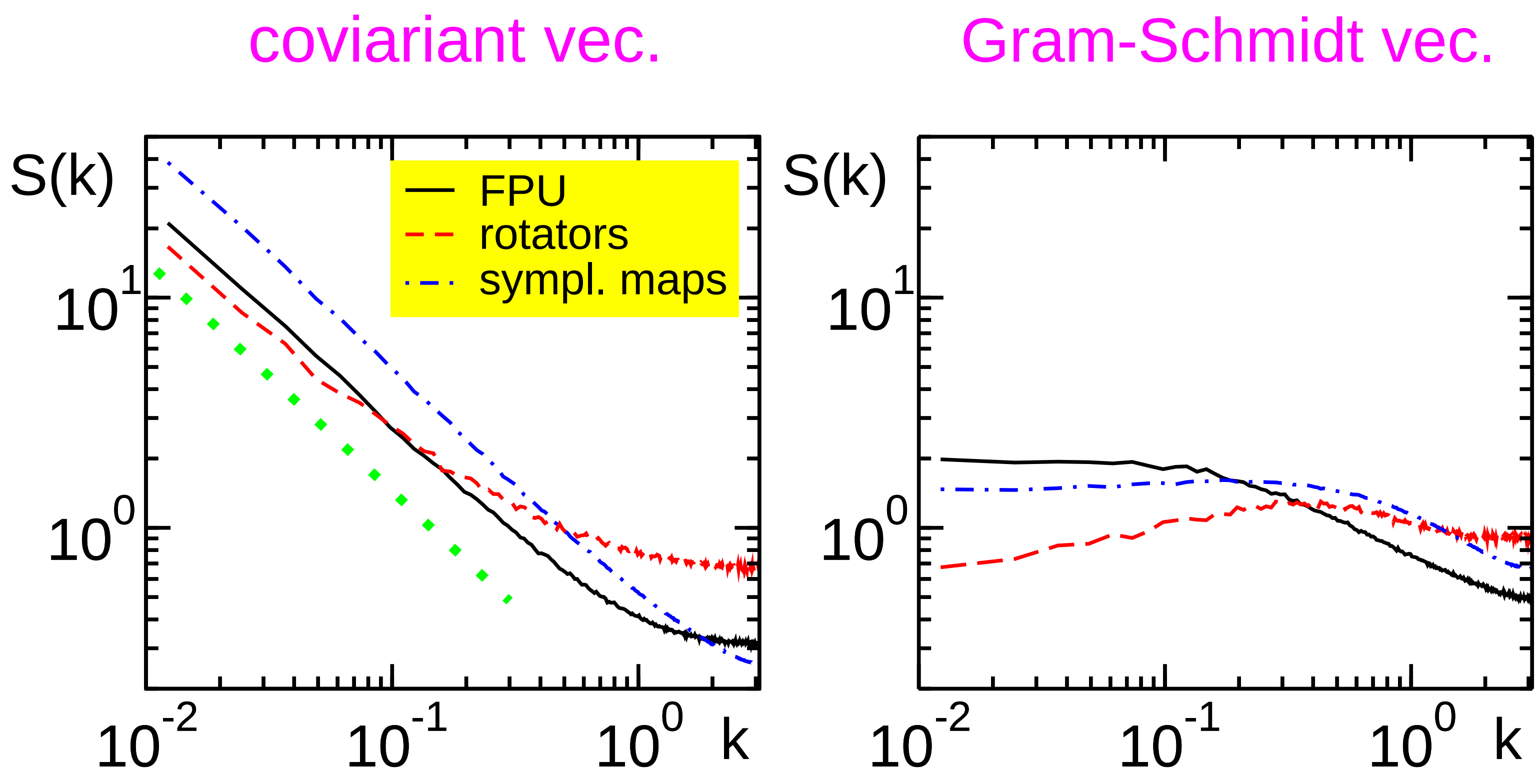
h=0.2



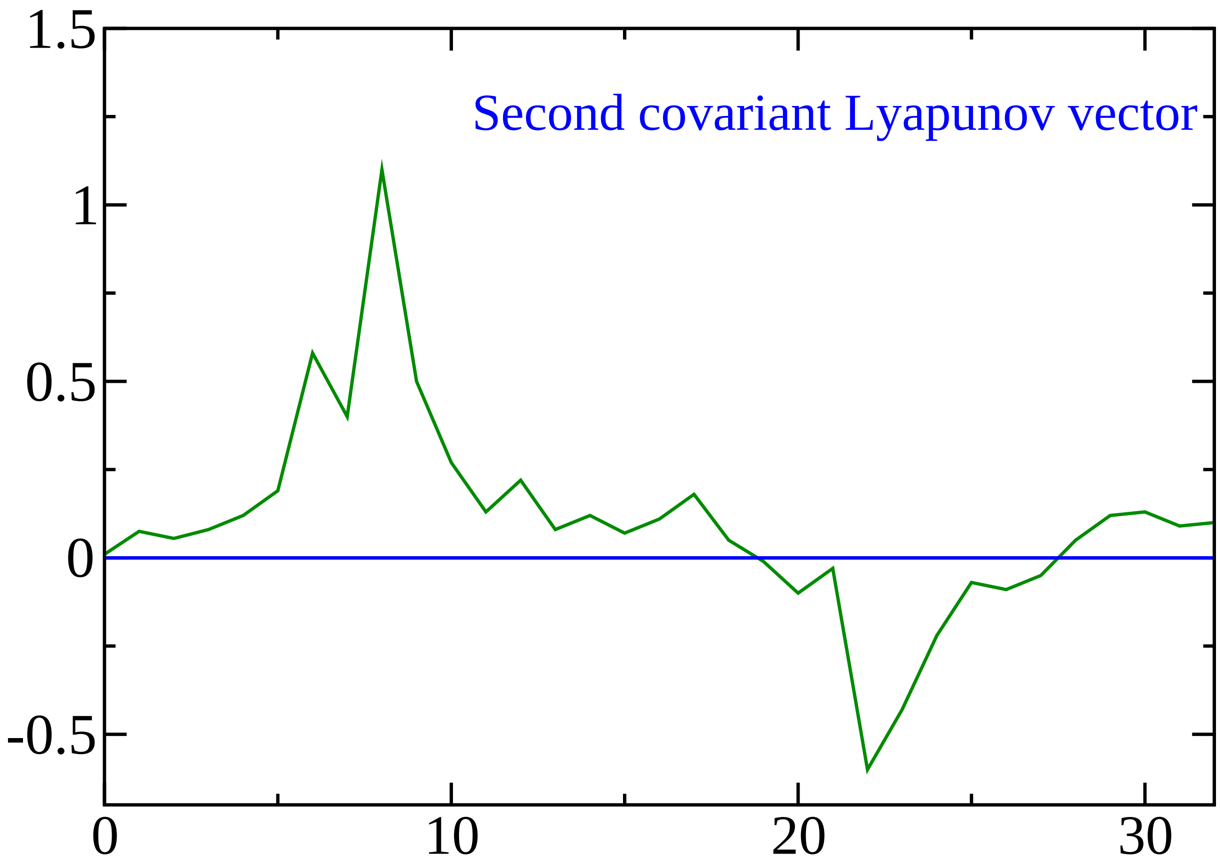
Asymptotic localization



FOURIER ANALYSIS OF THE "HYDRODYNAMIC" MODES
THE LEAST EXPANDING DIRECTION



A.P., A. Torcini, and S. Lepri, J. Physique (1998)



J.M. Lopez, D. Pazo et al. (in preparation)

COLLECTIVE BEHAVIOUR IN GLOBALLY COUPLED SYSTEMS: STUART-LANDAU OSCILLATORS

$$\dot{z}_j = z_j - (\alpha + i\beta)|z_j|^2 z_j + \langle z_j \rangle$$

Global modes can be inferred from the covariant Lyapunov vectors

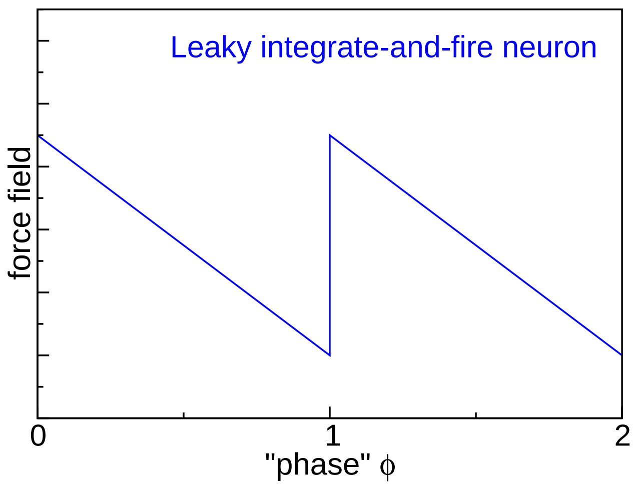
`Extended vectors = collective modes`

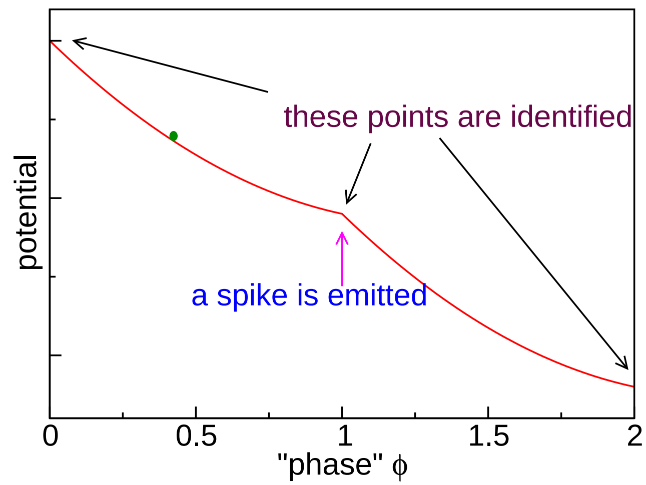
`Localized vectors = microscopic dynamics`

(Chaté Ginelli, preliminary results)

PULSE-COUPLED LEAKY INTEGRATE-AND-FIRE NEURONS

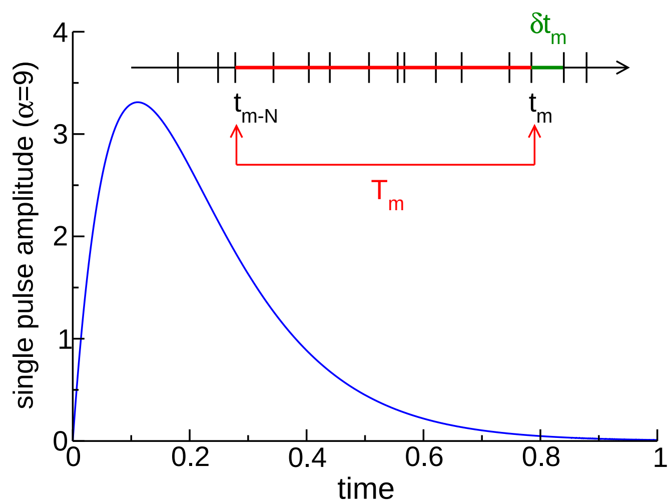
$$\dot{\phi}_i = a + \lambda(1-\phi_i) + gE(t)$$



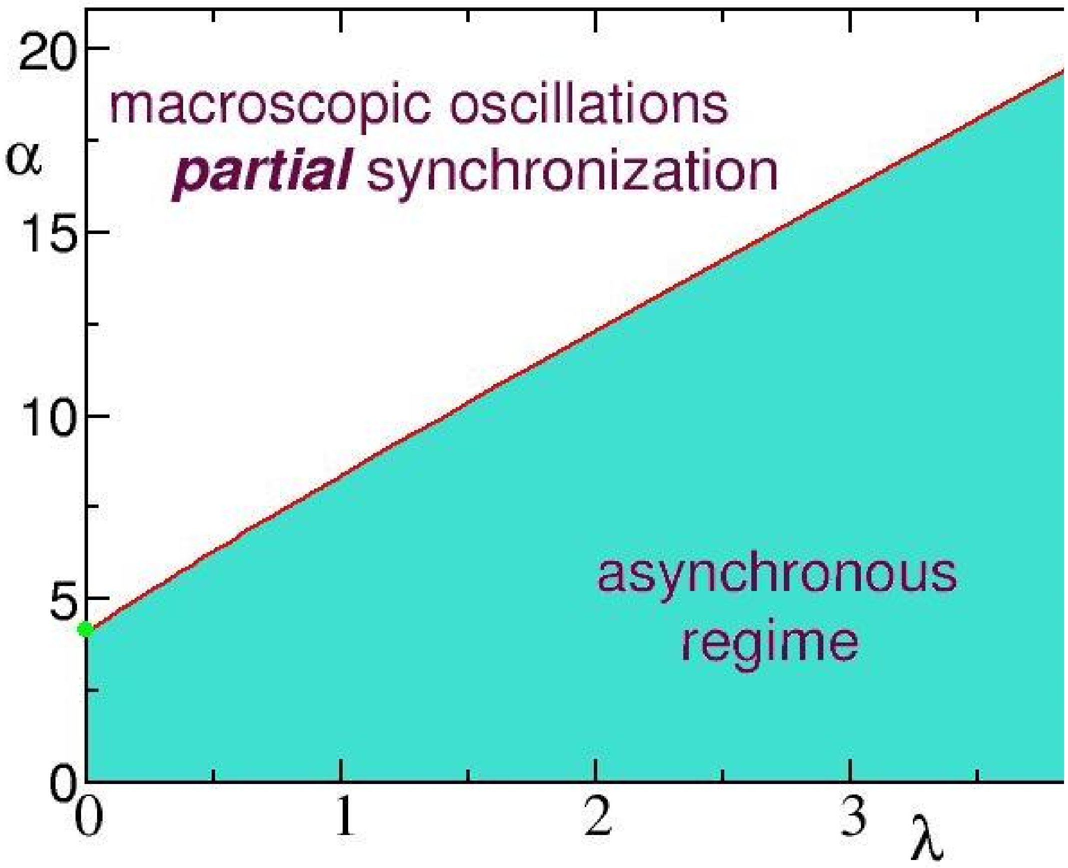


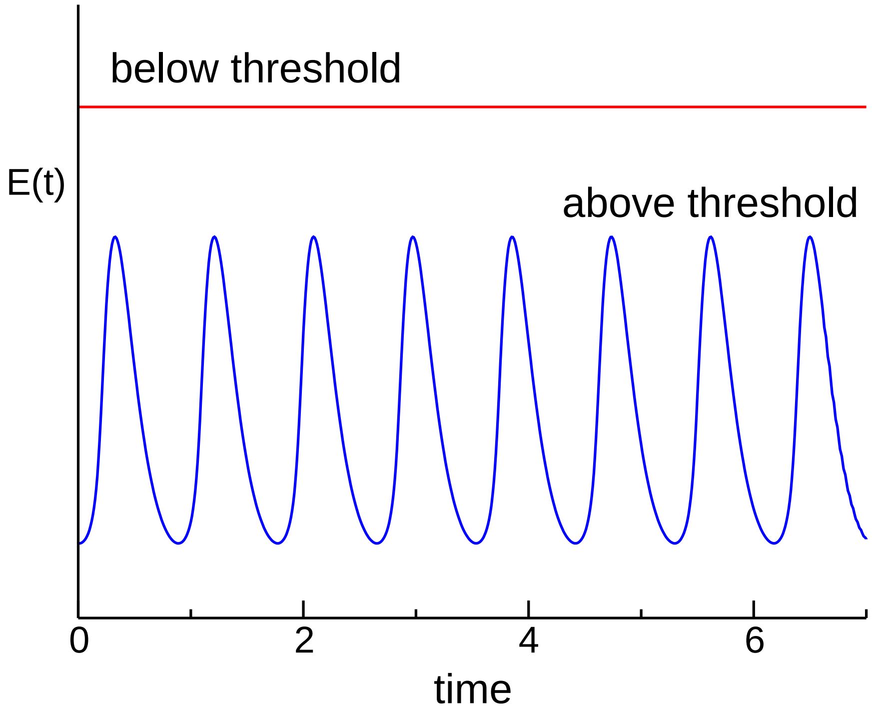
$$E(t) = \frac{\alpha^2}{N}\sum_m (t-t_m)\mathrm{e}^{-\alpha(t-t_m)}$$

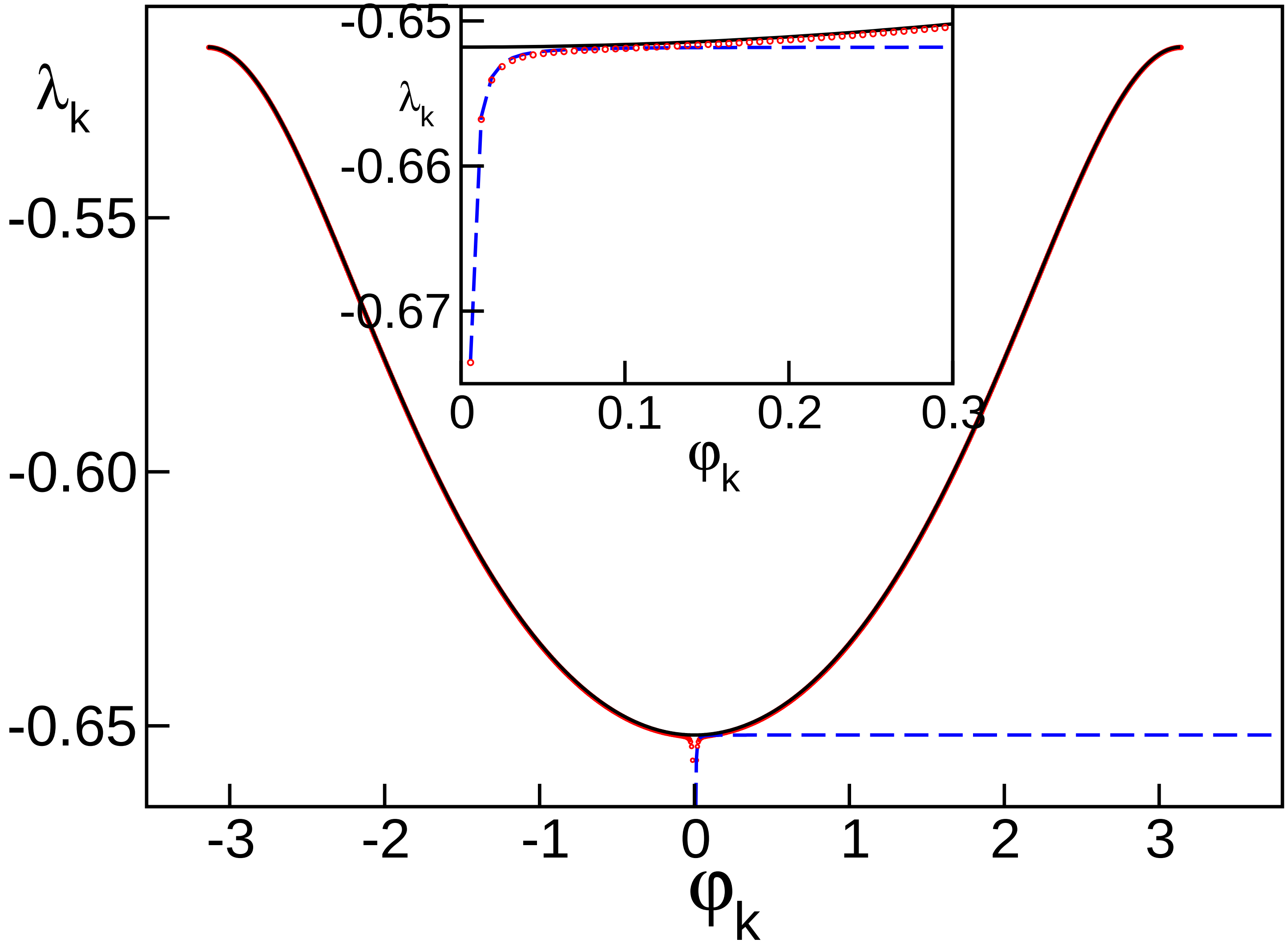
[C. van Vreeswijk, PRE, **54** 5522 (1996)]



MACROSCOPIC PERIODIC OSCILLATIONS





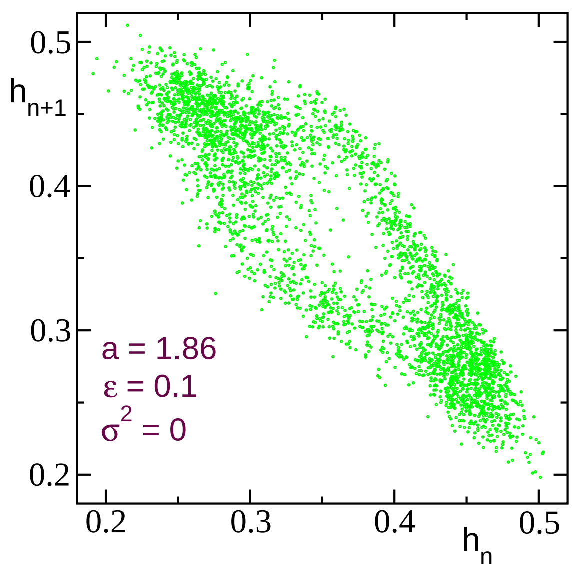


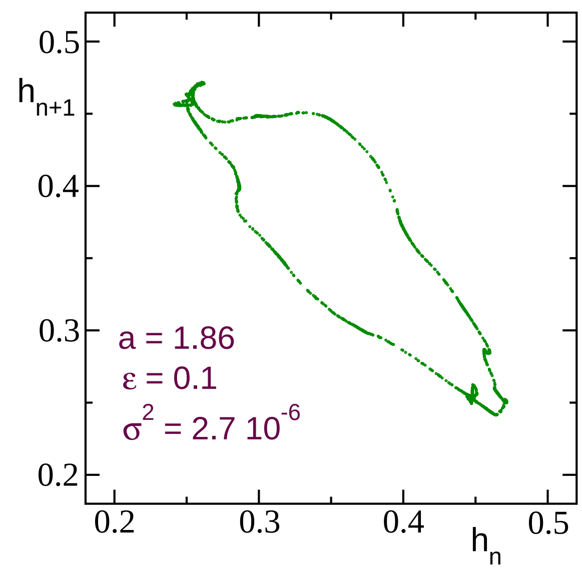
-0.65
-0.66
-0.67
λ_k
0
0.1
0.2
0.3
φ_k
-0.55
-0.60
-0.65
-3
-2
-1
0
1
2
3

GLOBALLY COUPLED MAPS

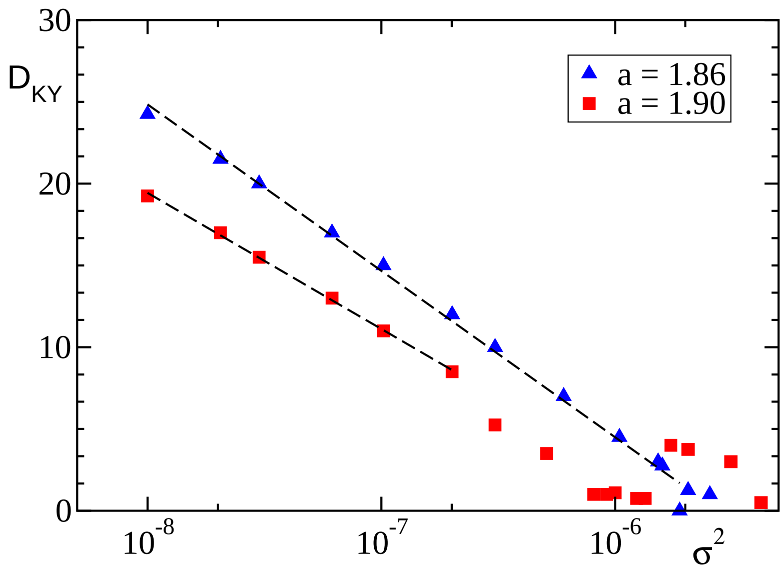
$$x_{n+1}^i = (1-\varepsilon)f(x_n^i) + \frac{\varepsilon}{N}\sum_{j=1}^N f(x_n^j) + \xi_n$$

$$f(x) = 1 - ax^2 \qquad\qquad h_n := \frac{\varepsilon}{N}\sum_{j=1}^N f(x_n^j)$$



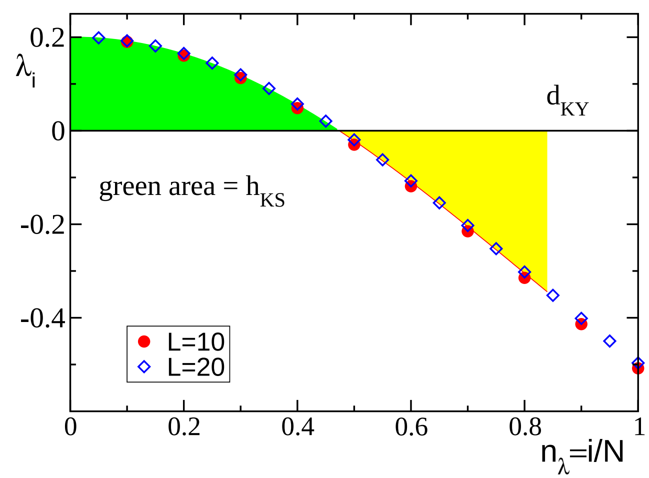


[T. Shibata, T. Chawanaya, K. Kaneko, PRL **82** 1999]



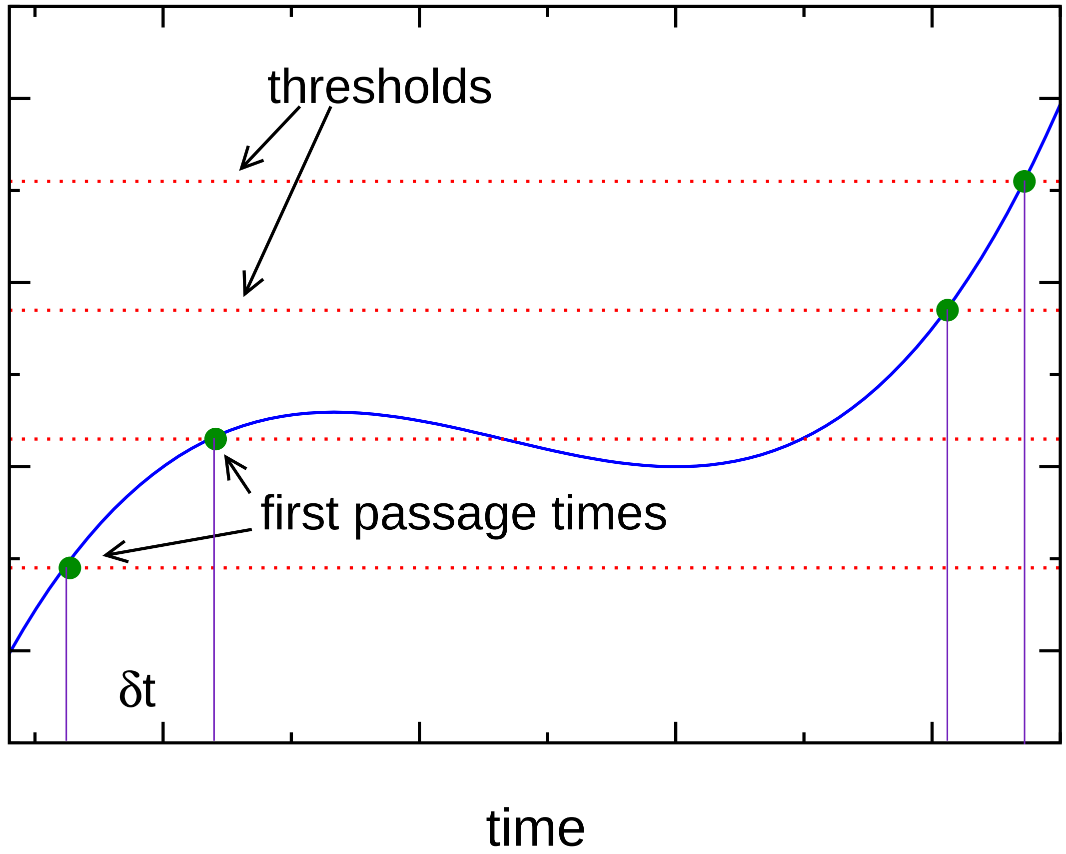
$$\rho_{n+1}(x) = \int dy \frac{\rho_n(y)}{\sqrt{2\pi}\sigma} \mathrm{e}^{-[F_n(y)-x]^2/2\sigma^2}$$

$$F_n(x) = (1-\varepsilon) f(x) + \varepsilon h_n$$



$$D_{KY} = M + \frac{\sum_{i=1}^{M} \lambda_i}{|\lambda_{M+1}|}$$

FINITE-AMPLITUDE LYAPUNOV EXPONENT(s)



$$\lambda(\Delta) = \left\langle \frac{\ln \Delta}{\ln \delta t(\Delta)} \right\rangle$$

[M.Cencini, M.Falcioni, D.Vergni, A.Vulpiani, Phys.D **130** 58 (1999)

F.Ginelli, R.Livi, A.P., A.Torcini, PRE **67** 046217 (2003)

$$x^i_{n+1} = (1-\varepsilon) f(x^i_n) + \frac{\varepsilon}{N} \sum_{j=1}^{N} f(x^j_n) + \xi_n$$

$$f(x) = 1 - ax^2$$

$$h_n := \frac{\varepsilon}{N} \sum_{j=1}^{N} f(x^j_n)$$