Route to chaos for piecewise smooth systems

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Introduction

- Poincar map
- Period doubling and route to chaos
- Corner bifurcation
- Sliding bifurcation
- Grazing bifurcation
- Implicit function theorem

Recalls and Problem statement:

Systems submitted to sliding bifurcations:

Let us consider the following piecewise smooth system:

$$\dot{x} = \begin{cases} F_1(x) \text{ if } H(x) \geq 0 \\ F_2(x) \text{ if } H(x) < 0 \end{cases} \tag{0.1}$$

where $x : I \longrightarrow D$, D is an open bounded domain of R^n with $n \geq 3$, $I \subset R^+$, generally I is the time interval.

$$F_1, F_2 : C_{abs}(I, D) \longrightarrow C^k(I, D), \qquad \text{for} \quad k \geq 4$$

where $C^k(I, D)$ is the set of C^k functions defined on I and having values in R^n,the norm on $C^k(I, D)$ is defined as follows:

$$\|x\|_k = \sup_{t \in I} \|x(t)\|_n + \sup_{t \in I} \|\dot{x}(t)\|_n + ... + \sup_{t \in I} \left\|x^{(k)}(t)\right\|_n$$

$C_{abs}(I,D)$ is the set of absolutely continuous functions defined on I and having values in D provided with the norm:
$||x|| = \sup_{t\in I} ||x(t)||_n$.

According to (Bresis, 1999), $(C^k(I,D), \|.\|_k)$ and $(C_{abs}(I,D), \|.\|)$ are two Banach spaces.

$H : D \longrightarrow R$ is a C^{1}application that defines :Textcolorblue

$$S = \{x(t) \in D : H(x(t)) = 0\}$$

$$S^+ = \{x(t) \in D : H(x(t)) > 0\}$$

$$S^- = \{x(t) \in D : H(x(t)) < 0\}$$

It is assumed that there exists a subset of switching manifold $\bar{S} \subset S$ that denotes a sliding region which is simultaneously attracting from $S^+$and S^-

H^s-1) $< \nabla H(x(t)), F_2(x(t)) - F_1(x(t)) >> 0$, $\forall x(t) \in v_{\bar{S}}$

Under H^s-1), if the system trajectory crosses $\bar{S}$, the sliding motion evolves in $\bar{S}$ until it eventually reaches its boundary. Now, it is assumed that the system (0.1) depends smoothly on a parameter ε such that at $\varepsilon = 0$, there exists a periodic orbit $x(t)$ that slides at the point x^* corresponding to t^*:

$$\dot{x} = \begin{cases} F_1(x, \varepsilon) \text{ if } H(x) \geq 0 \\ F_2(x, \varepsilon) \text{ if } H(x) < 0 \end{cases} \tag{0.2}$$

H^s-2) $H(x^*) = 0$ and $\nabla H(x^*) \neq 0$.

H^s-3) $< \nabla H(x^*(t)), F_1(x^*)) > = 0$

So, without loss of generality (thank to a coordinate change), the bifurcation point is assumed to be located at $(x^*, t^*) = (0, 0)$ and according to (di Bernardo & al., 2002), four possible cases are possible:

Case 1: Sliding bifurcation of type 1:

A^s-1) $\frac{\partial^2 H(\phi_1(0,0))}{\partial t^2} = \left\langle \nabla H(x^*(0)), \frac{\partial F_1(x^*(0))}{\partial x} F_1(x^*(0)) \right\rangle > 0$

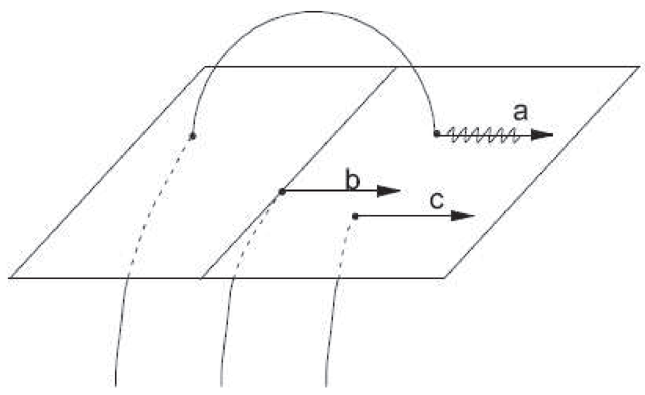

Figure 1: Sliding bifurcation type 1

Case 3: Sliding bifurcation type 2:

A^s-3) $\frac{\partial^2 H(\phi_1(0,0))}{\partial t^2} = \left\langle \nabla H(x^*(0)), \frac{\partial F_1(x^*(0))}{\partial x} F_1(x^*(0)) \right\rangle < 0.$

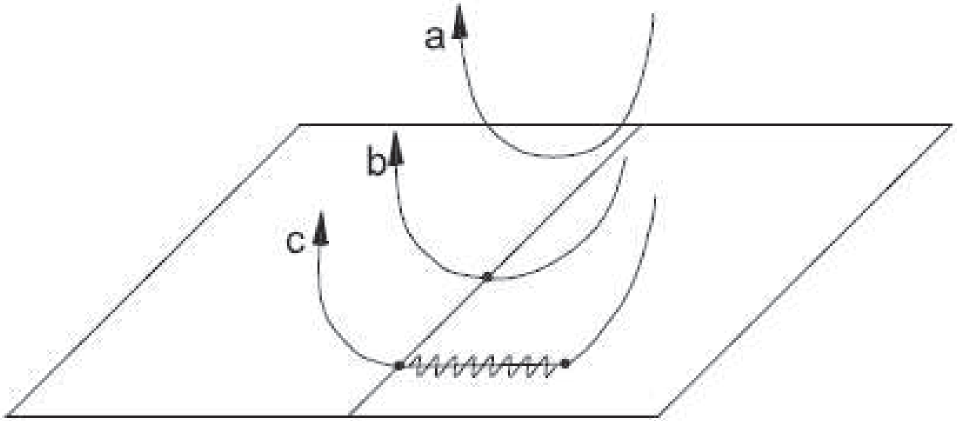

Figure 3: Sliding bifurcation type 2

Case 2: Grazing sliding bifurcation:

A^s-2): $\frac{\partial^2 H(\phi_2(0,0))}{\partial t^2} = \left\langle \nabla H(x^*(0)), \frac{\partial F_2(x^*(0))}{\partial x} F_2(x^*(0)) \right\rangle > 0.$

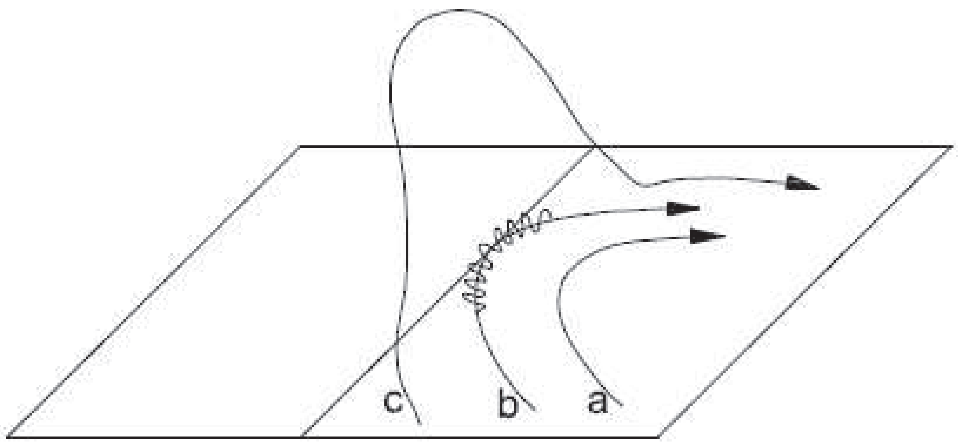

Figure 2: Grazing Sliding bifurcation

Case 4: Multisliding bifurcation:

A^s-4) $$\frac{\partial^3 H(\phi_1(0,0)}{\partial t^3} = \left\langle \nabla H(x^*(0)), (\frac{\partial F_1(x^*(0))}{\partial x})^2 F_1(x^*(0)) \right\rangle < 0.$$

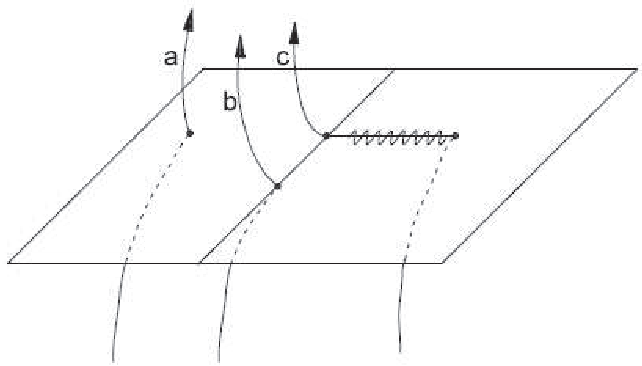

Figure 4: Multisliding Sliding bifurcation

Its corresponding normal form is :

$$P_4(x,\varepsilon) = \begin{cases} \varepsilon x & \text{if} < \nabla H, x > \geq 0 \\ \varepsilon x + \varepsilon^2 v_4(x) + o(\varepsilon^{\frac{5}{2}}) & \text{if} < \nabla H, x > < 0 \end{cases}$$

where:

$$\begin{aligned} v_4(x) \quad = \quad & -\frac{9}{2}\frac{< \nabla H, \frac{\partial F_1}{\partial x} x >^2}{< \nabla H, F_2 >< \nabla H, (\frac{\partial F_1}{\partial x})^2 x >}(\frac{\partial F_1}{\partial x} \\ & -\frac{< \nabla H, \frac{\partial F_1}{\partial x} F_2 >}{< \nabla H, F_2 >} I_d)(F_2 - F_1) \end{aligned} \tag{0.3}$$

Lemma 1

If $(<\nabla H, x> > 0$ *for case 1 and 3) or* $(<\nabla H, x> < 0$ *for case 2 and 4), then the problem of finding a periodic solution of problem (0.2) is equivalent to analyze for each type of sliding bifurcations the following corresponding equations:* $\beta_i(x, \varepsilon) - x = 0$, *where:*

- *For case 1:* $\beta_1(x, \varepsilon) = \varepsilon x + \varepsilon^2 v_1(x) + o(\varepsilon^3)$,
- *For case 2:* $\beta_2(x, \varepsilon) = \varepsilon x + \varepsilon v_2(x) + o(\varepsilon^{\frac{3}{2}})$,
- *For case 3:* $\beta_3(x, \varepsilon) = \varepsilon x + \varepsilon^3 v_3(x) + o(\varepsilon^4)$,
- *For case 4:* $\beta_4(x, \varepsilon) = \varepsilon x + \varepsilon^2 v_4(x) + o(\varepsilon^{\frac{5}{2}})$,

Systems submitted to Grazing bifurcations:

- A^g-1) $H(0) = 0$ and $\nabla H(0) \neq 0$.
- A^g-2)$< \nabla H(0), \frac{\partial \Phi_i}{\partial t}(0,0) >=< \nabla H(0), F_i^0 > = 0, i = 1, 2.$
- A^g-3) for $i = 1, 2$.

$$\begin{aligned} \frac{\partial^2 H(\Phi(0,0))}{\partial t^2} &= (< \nabla H(0), \frac{\partial F_i^0}{\partial x} F_i^0 > \\ &+ < \frac{\partial^2 H(\Phi(0,0))}{\partial x^2} F_i^0, F_i^0 >) > 0 \end{aligned}$$

- A^g-4) $$(< L, F_1^0 >< L, F_2^0 >) > 0$$

where Φ_i are the flows associated to F_i , $F_i^0 = F_i(\Phi_i(0,0))$, $i = 1, 2$ and L is the unit vector perpendicular to $\nabla H(0)$.

The Poincaré map is given by:

$$P(x,\varepsilon) = \begin{cases} P_1(x,\varepsilon) & \text{if} < \nabla H, x > \quad > 0 \\ P_2(x,\varepsilon) & \text{if} < \nabla H, x > \quad < 0 \end{cases}$$

Where:

$$\begin{aligned} P_1(x,\varepsilon) &= Nx + M\varepsilon + o(\|x\|_{n-1}, \varepsilon) \\ P_2(x,\varepsilon) &= N(x + v_1| < \nabla H, x > |)^{\frac{3}{2}} \\ &\quad + v_3(< \nabla H, \frac{\partial F_2}{\partial x} x >)(| < \nabla H, x > |)^{\frac{1}{2}} \\ &\quad + v_2 x(| < \nabla H, x > |)^{\frac{1}{2}} + M\varepsilon + o(\|x\|^2_{n-1}, \varepsilon)) \end{aligned} \tag{0.4}$$

N is a nonsingular matrix $(n-1)\times(n-1)$, M is a nonzero $n-1$ dimensional vector.

Lemma 2

Searching a periodic solution of (0.2) at grazing is equivalent to resolve the following equation:

$$P_2(x, \varepsilon) - x = 0$$

where $P_2(x, \varepsilon)$ *is given by (0.4).*

SYSTEMS SUBMITTED TO CORNER BIFURCATIONS:

For this case the following piecewise smooth system is required:

$$\dot{x} = \begin{cases} F_1(x) \text{ if } H_1(x) \geq 0 \text{ or } H_2(x) \geq 0 \\ F_2(x) \text{ if } H_1(x) < 0 \text{ and } H_2(x) < 0 \end{cases} \quad (0.5)$$

for $i = 1, 2,\ H_i : D \longrightarrow R$ are assumed to be $C^1$and define the sets: $S_1 = \{x(t) \in D : H_1(x(t)) = 0\}$ and $S_2 = \{x(t) \in D : H_2(x(t)) = 0\}$ such that S_1 and S_2 intersect along a corner C with a smooth codimension two surfaces and the following regions are not empty:

$$\begin{aligned} D_{int} &= \{x(t) \in D : H_1((x(t)) < 0 \text{ and } H_2((x(t)) < 0\} \\ D_{out} &= \{x(t) \in D :\ H_1(x(t)) \geq 0 \text{ or } H_2(x(t)) \geq 0\} \end{aligned}$$

This generates a non-zero angle, so, let's consider that $(0,0)$ is on the corner C:

H^c-1) $\nabla H_1(0,0) \times \nabla H_2(0,0) \neq 0$.

F_1 and F_2 are assumed to be defined on both sides of D_{int} and D_{out} respectively.

- Assumption for the external corner collision:

A^c-1)$< F_i(0,0), \nabla H_1(0,0)) >< 0$ and $< F_i(0,0), \nabla H_2(0,0) >> 0$, $i = 1,2$.

- Assumption for the internal corner collision:

A^c-2)$< F_i(0,0), \nabla H_1(0,0)) > \geq 0$ or $< F_i(0,0), \nabla H_2(0,0) > \leq 0$, $i = 1,2$.

As for the previous case and in order to present a complete bifurcation analysis, one assumes that the system (0.5) depends smoothly on a parameter ε:

$$\dot{x} = \begin{cases} F_1(x,\varepsilon) \text{ if } H_1(x) \geq 0 \text{ or } H_2(x) \geq 0 \\ F_2(x,\varepsilon) \text{ if } H_1(x) < 0 \text{ and } H_2(x) < 0 \end{cases} \tag{0.6}$$

Case 1 The considering trajectory enters in D_{int} in the neighborhood of C, the Poincaré normal form is given by :

$$P_1(x,\varepsilon) = \begin{cases} Px + Q\varepsilon + o(x,\varepsilon) \;\text{ if }\; \text{non-crossing} \\ Px + (F_1 - F_2) < a_2,\; x > +Q\varepsilon + o(x,\varepsilon) \\ \;\text{if crossing} \end{cases}$$

where:

$$a_2 = J_2 - \frac{1}{2} < J_2, F_1 > J_1, J_i = \frac{\nabla H_i}{< \nabla H_i, F_i >}, i = 1,2 \tag{0.7}$$

Case 2 External bifurcation: the corresponding Poincaré normal form is given by:

$$P_2(x,\varepsilon) = \begin{cases} Px + Q\varepsilon + o(x,\varepsilon) \text{ if non-crossing} \\ Px + (F_1 - F_2) < a_1, x > +Q\varepsilon + o(x,\varepsilon) \\ \quad \text{if crossing} \end{cases}$$

where

$$a_1 = J_1 - < J_1, F_2 > J_2 \tag{0.8}$$

Lemma 3

When a crossing occur, the problem of finding a periodic solution for problem (0.6) is equivalent to analyze for each case $i = 1, 2$, the following corresponding equations:

$$\beta_i(x, \varepsilon) - x = 0$$

- *for case 1, (internal collision-bifurcation):*

$$\beta_1(x, \varepsilon) = Px + (F_1 - F_2) < a_2,\ x > +Q\varepsilon + o(x, \varepsilon)$$

where a_2 *is given by (0.7) and* $(x, \varepsilon)\epsilon V_x \times V_\varepsilon$.

- *for case 2, (external collision-bifurcation):*

$$\beta_2(x, \varepsilon) = \quad Px + (F_1 - F_2) < a_1,\ x > +Q\varepsilon + o(x, \varepsilon)$$

where a_1 *is given by (0.8) and* $(x, \varepsilon)\epsilon V_x \times V_\varepsilon$.

Route to Chaos:

Precisely, the problem becomes to determine for each map β_i ,$i = 1, 2$, three distinct points noted respectively x_i, y_i and z_i such that:

$$\beta_i(x_i, \varepsilon) = y_i, \qquad \beta_i(y_i, \varepsilon) = z_i \qquad \text{and} \qquad \beta_i(z_i, \varepsilon) = x_i$$

This well be done naturally in three steeps as follows and thank to Implicit unction theorem.

Sliding bifurcation case

Corollary 1

Under conditions H^s-j), $j = 1, 2, 3,\ 4, 5,$ and A^s-i), $i = 1, 2, 3, 4$ specific to each case, the system (0.2) of dimension 3 admits a chaotic behavior at sliding collision.

H^s-4) and H^s-5) are necessaries conditions for invoking implicit function theorem.

Grazing bifurcations case:

In this case we have some mathematical computations:

A^g-5) $(Nx + M\varepsilon - x)^T (N(v_1(\delta(x))^2 + v_2 x(\delta(x))$
$+v_3(< \nabla H, \frac{\partial F_2}{\partial x} x >)(\delta(x))) + o(\|x\|^2, \varepsilon)) > 0$

A^g-6) $\frac{\partial \Gamma_i^g}{\partial \eta_1}(0,0,0) \neq 0.$

A^g-7) $\frac{\partial \Pi_i^g}{\partial \eta_2}(0,0) \neq 0.$

and we obtain

Corollary 2

Under assumptions A^g-i) for i= 1, 2, 3, 4, 5, 6,7, the system (0.5) admits a chaotic behavior at grazing collision.

Corner bifurcations case:

- A^c-3)$(P - I_d)$ is a nonsingular matrix, where I_d is the $(n-1) \times (n-1)$ identity matrix.
- A^c-4,i) $B_i = (P - I_d + (F_1 - F_2)a_i^T)$ is a nonsingular matrix.
- A^c-5,i)$C_i = (P - 2I_d + (F_1 - F_2)a_i^T)$ is a nonsingular matrix.
- A^c-6,i)$D_i = (B_i\widetilde{C}_j + I_d^0) - \widetilde{C}_j - I_d^0$) is a vector such that it's j component (noted d_j) is non null.
- A^c-7,i) $E_i = B_i(\widetilde{C}_j\alpha + \alpha I_d^0 + I_d^0) - \widetilde{C}_j\alpha$ is a non null scalar, where $\alpha = -\frac{1}{d_j}, i = 1, 2.$

One obtains:

Corollary 3

Under conditions H^c-1), A^c-1) A^c-2), A^c-3,i), A^c-4i), A^{c}5-i), A^{c}6-i) and A^{c}7-i) specific to each case $i = 1, 2$, the system (0.6) admits a chaotic behavior at corner collision.

AN ILLUSTRATIVE EXAMPLE:

Let us consider the following piecewise smooth system:

$$\begin{cases} \dot{x}_1 = f_{11}(x_1, x_2, x_3) \\ \dot{x}_2 = f_{12}(x_1, x_2, x_3)) \\ \dot{x}_3 = f_{13}(x_1, x_2, x_3) \end{cases} \quad if\ x \epsilon D_{int} \tag{0.9}$$

and

$$\begin{cases} \dot{x}_1 = f_{21}(x_1, x_2, x_3) \\ \dot{x}_2 = f_{22}(x_1, x_2, x_3) \\ \dot{x}_3 = f_{23}(x_1, x_2, x_3) \end{cases} \quad if\ x \epsilon D_{out} \tag{0.10}$$

where:

$$
\begin{aligned}
f_{11} &= x_2 - (x_1^2 + x_2^2 - (1 + 10\varepsilon(\sin(777x_3) + 1)^2))x_1 \\
f_{12} &= -x_{1-}(x_1^2 + x_2^2 - (1 + 10\varepsilon(\sin(777x_3) + 1)^2))x_2 \\
f_{13} &= \varepsilon(x_3 - \cos(330x_2) - (x_3 - \sin x_2)^3)
\end{aligned}
$$

$$
\begin{aligned}
f_{21} &= \tfrac{x_2}{\pi} - 100x_3^2((x_1 + 2)^2 + x_2^2 - (3 + 10x_3)^2)x_1 \\
f_{22} &= -(\tfrac{x_1+2}{\pi}) - 100x_3^2((x_1 + 2)^2 + x_2^2 - (3 + 10x_3)^2)x_2 \\
f_{23} &= 10\pi\varepsilon(x_3 + \sin(500x_1x_3)) - \varepsilon x_3^3
\end{aligned}
$$

$S_1 = \left\{(x_1, x_2, x_3)\epsilon\mathbb{R}^3 : H_1(x_1, x_2, x_3) = -x_1 + 1\right\}$
$S_2 = \left\{(x_1, x_2, x_3)\epsilon\mathbb{R}^3 : H_2(x_1, x_2, x_3) = -x_2\right\}$
The conditions for internal collision are satisfied and computations give us the corresponding Poincaré map:

$$P(x,\varepsilon) = \begin{cases} \begin{pmatrix} e^{2\pi} & 0 \\ 0 & -e^{-2\pi} \end{pmatrix} x + o(x,\varepsilon) \quad \text{if} \quad \text{non-crossing} \\ \left(\begin{pmatrix} e^{2\pi} & 0 \\ 0 & -e^{-2\pi} \end{pmatrix} x + (-0.231, -0.113)^T \right. \\ \left. < (2.411, -1.013)^T,\ x > + o(x,\varepsilon) \right) \\ \text{if} \quad \text{crossing} \end{cases}$$

where $x = (x_1, x_2)^T$ and $o(x,\varepsilon) \longrightarrow 0$ when $(x,\varepsilon) \longrightarrow (0,0)$.

For all simulations, the system is initialized in $x_1(0) = 1.1$, $x_2(0) = 0$, $x_3(0) = 0$. Moreover, in order to highlight the proposed way to chaos different values of ε in zero's vicinity are considered:

- For $\epsilon = 0$, the system (0.9 and 0.10) generates an attractive limit cycle see figure 1.
- For $\epsilon = 0.001$, a period doubling appears see figure 2.
- For $\epsilon = 0.1$, a chaotic behavior occurs see figure 3.

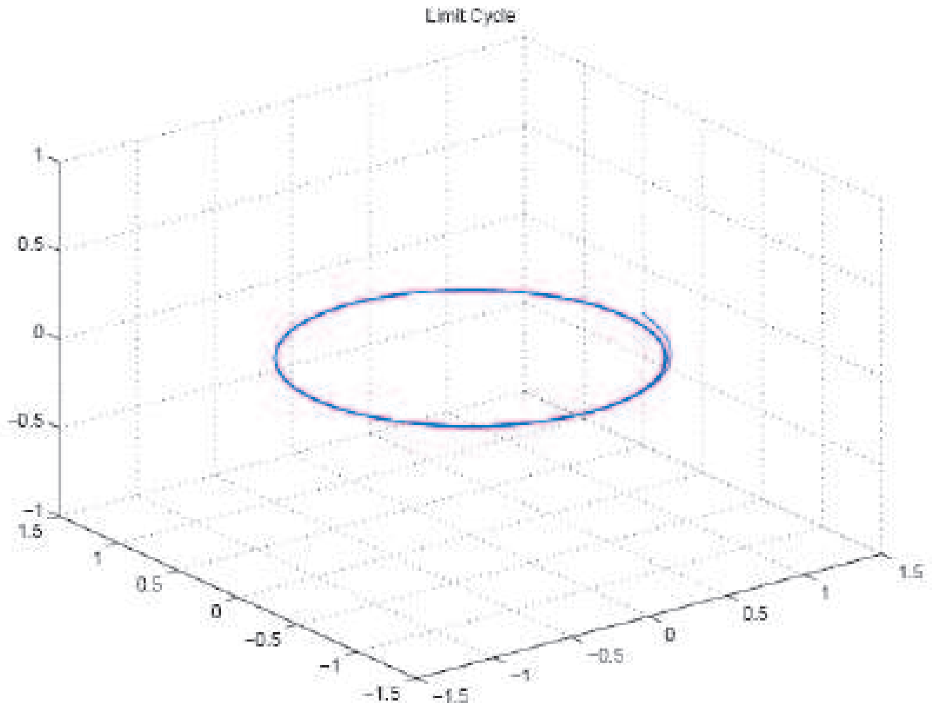

Figure 1: Figure 1: $\epsilon = 0$

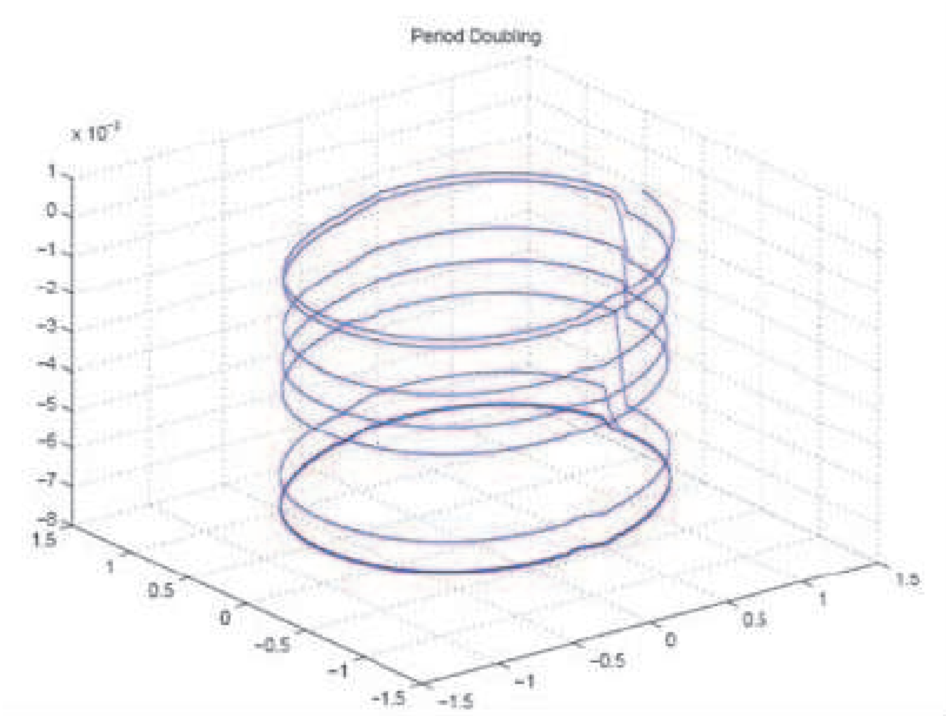

Figure 2: Figure 2: $\epsilon = 0.001$

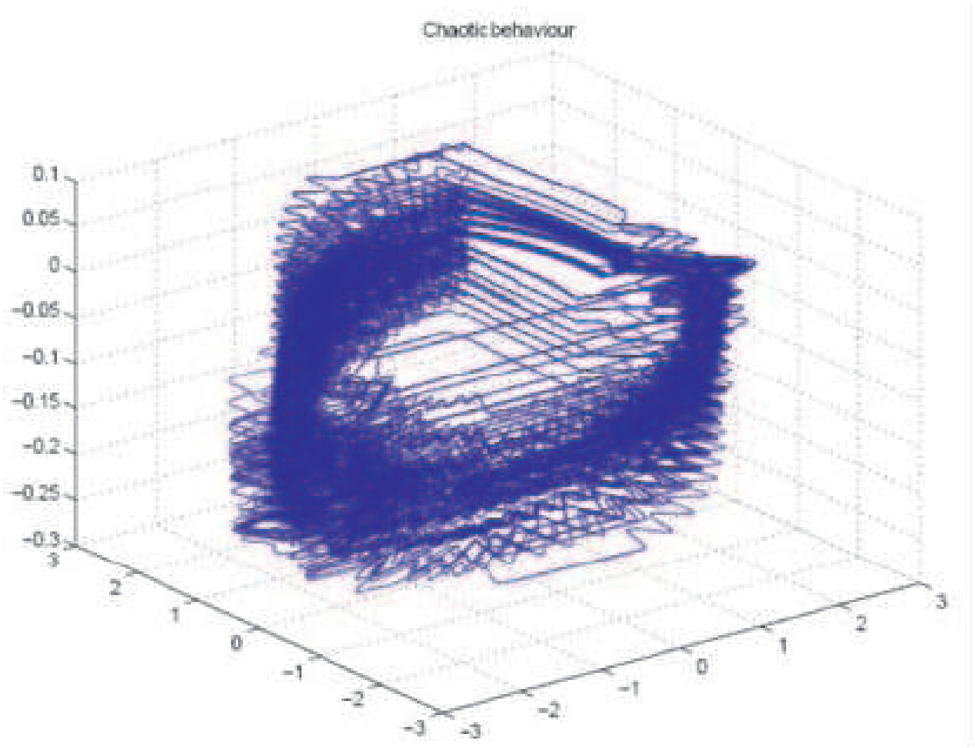

Figure 3: Figure 3: $\epsilon = 0.1$

Conclusion

- This approach shows that route to chaos is available not only for some fixed value of the bifurcation parameter but also for any value of this parameter in some small neighborhood of this value.
- Note that a mathematical tolls as Implicit Function Theorem, Poincaré map and period doubling are enough for analyze chaotic behavior of piecewise smooth systems.

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