Delay systems
Batz-sur-Mer October 17-21 2011

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PLAN

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9. Square-waves (Laurent Larger)
10. Control theory (Jean-Pierre Richard)
1. Familiar delay problems

- The hot shower problem

\[ \frac{dT(t)}{dt} = -a(T(t - \tau) - T_d) \]

\[ a = T_d = \tau = 1 \]

\[ T = 0.5 \quad (-\tau < t \leq 0) \]

**Delay:** time for the increased (or decreased) hot/cold water to flow from the tap to the shower head
\[
\frac{dT(t)}{dt} = -a(T(t - \tau) - T_d)
\]
\[a = T_d = \tau = 1\]
\[T = 0.5 \quad (-\tau < t \leq 0)\]

\[
\tau = \frac{8\mu l}{\Delta p R}
\]

\[a\tau > e^{-1} \approx 0.36 : \text{damped oscillations}\]

\[a\tau > \pi / 2 \approx 1.57 : \text{growing oscillations}\]

\[a \text{ large: high human sensitivity}\]

\[\tau \text{ large: low } \Delta p, \text{ large } l\]
• Remote control
Images are sent to Earth and a signal is sent back

For the Moon, the time delay in the control loop is 2-10 s
For Mars, it is 40 minutes!
• Traffic flow

\[ a_2(t + \tau) = \alpha(v_1 - v_2) \]

↑ braking speed difference

\( \tau \): reaction time
\[ a_2(t + \tau) = \alpha(v_1 - v_2) \]
\[ v_2'(t + \tau) = \alpha(v_1 - v_2) \]

\[ \alpha = 0.5 \text{ s}^{-1}, \quad \tau = 1 \text{ s} \]
\[ v_1(\text{km/h}) = 80 + 20(t - \tau) \quad (\tau \leq t < 0) \]

\[ d' = v_1 - v_2 \]
\[ a_2(t + \tau) = \alpha(v_1 - v_2) \]

\( \tau \): reaction time

sober driver

\( \tau = 1 \text{s} \)

0.5g/l in blood

= 1 shot

= 2 glasses of wine

= 2 cans of beer

\( \tau = 1.5 \text{s} \)
2. Emerging problems

- Wind instruments such as the clarinet

Researchers (J.-P. Dalmont Université du Maine) et J. Kergomard (Université d’Aix-Marseille) study how it works.

Goal: synthesize the sound of a clarinet or a trumpet

Delay = round trip of the pressure wave
• Neuron synchronization (Parkinson disease)

Tremors result from a too synchronized population of neurons

What is the role of the delayed communication between neurons?

Researchers J. Henry (INRIA Bordeaux)
A. Beuter, J. Modolo (Université Bordeaux 2)
model networks of coupled neurons
• **Anticipation mechanisms**
The reflex time is too long. Researchers think that the brain develop anticipation mechanisms to compensate for the delay.

\[
\text{signal hand – brain} = 0.1 \text{ s} \\
\text{signal round trip time} = 0.2 \text{ s} \\
\text{Too long!}
\]

For example, our hand is preparing itself to catch the ball
3. Delay Equation?

\[
\frac{dy(t)}{dt} = -ky(t) \quad \rightarrow \quad y = A \exp(-kt)
\]
3. Delay Equation?

\[ \frac{dy(t)}{dt} = -ky(t) \rightarrow y = A \exp(-kt) \]

\[ \frac{dy(t)}{dt} = -ky(t-\tau) \rightarrow y = A\sin(\omega t) \]

if \( \omega \tau = \frac{\pi}{2} \)
3. Delay Equation?

\[ \frac{dy(t)}{dt} = -ky(t) \rightarrow y = A \exp(-kt) \]

\[ \frac{dy(t)}{dt} = -ky(t - \tau) \rightarrow y = A \sin(\omega t) \]

if \( \omega \tau = \frac{\pi}{2} \)

Check

\[ A\omega \cos(\omega t) = -kA \sin(\omega(t - \tau)) \]

\[ \omega \cos(\omega t) = -k \left[ \sin(\omega t) \cos(\omega \tau) - \cos(\omega t) \sin(\omega \tau) \right] \]

(1) \( \omega = k \sin(\omega \tau) \) (2) \( \cos(\omega \tau) = 0 \)

\( \omega \tau = \pi / 2 \) and \( k = \omega \)
Summary

1. A time delay is taken into account when a mechanical or physiological feedback takes time.

2. Oscillatory regimes possible.
2009 – 2011

Population dynamics

- Density of lemmings (number of individuals per hectare) in the Churchill area in Canada (T.J. Case, Oxford 2002)

\[ \frac{dN}{dt} = rN (1 - N(t - \tau)) \]
\[ r = 1/(0.3 \text{ years}), \]
\[ \tau = 0.72 \text{ years} \]

Delay: the population growth depends on the present food levels.

Plant levels depends on the number of grazers that lived between now and \( \tau \) time steps earlier.

Broken line: solution of the delayed logistic equation.
The pupil eye reflex

(Beuter et al. Eds., Nonlinear Dynamics in Physiology and Medicine, Springer 2003)

When a narrow pencil of light is placed at the iris margin, a cycle of contraction and dilatation of the pupil is possible.

\[ \tau = 0.3 \text{ s} \]

P = 0.9 s useful for clinical tests
• Physiological diseases

• Sustained periodic breathing (PB) patients with cardiac problems sleep in altitude

\[
\frac{dp}{dt} = M - pV(p(t - \tau))
\]

↑ production  ↑ ventilation

CO₂ high in lungs
Blood carried the information to the brain
The brain commands ventilation of the lungs after a delay \( \tau \) (0.3 – 0.5 minutes)

• Blood disorders showing oscillatory clinical data
5. Engineering

Control air/fuel ratio (best 15:1 = 15 parts of air for one part of fuel)
• Mechanical engineering
  High speed machining
  (aeronautics, tool fabrication)

Danger: self-sustained vibrations = chatter
Machine-tool vibrations
chatter instabilities

Tool equation:
\[ y'' + \varepsilon y' + \omega_0^2 y = F \left[ y - y(t - \tau) \right] \]

\[ \omega_0 = 579 \text{ s}^{-1} \rightarrow T \approx 10^{-2} \text{s} \]
\[ \Omega = 14 \text{ turns s}^{-1} \rightarrow \tau = \Omega^{-1} = 7 \times 10^{-2} \text{s} \]
6. Optical feedback

- Semiconductor laser

\[ \tau_{\text{photons}} = 10^{-12} \text{s} \]

\[ \tau_{\text{round-trip}} = 10^{-9} \text{s} \]
6. Optical feedback

- Semiconductor laser

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\[ \tau_{\text{round-trip}} = 10^{-9} \text{s} \]

Large delay = oscillations = noise
Undesirable in optical communication
7. Stabilization with delay

\[ y'' + y = 0 \]

\[
y'' + y = y(t - \tau) - y
\]

delayed control

\[ = y - \tau y' + \ldots - y \]

\[ = -\tau y' + \ldots \]

\[ y'' + y + \tau y' = 0 \]

damping
Container cranes
Goal: move the container fast (24 tons)
Danger: oscillations

Idea: use a delayed control

\[
\frac{d^2 y}{dt^2} = F(y) + k(y(t - \tau) - y)
\]

↑
Newton
↑
delayed control

Results:

Oscillations are quickly damped when the control is ON

Model scale 1:10
for a 65 tons crane
Masoud and Daqaq
JJMIE 1, 57 (2007)
Summary

• Oscillatory instabilities caused by a delay occur in all scientific disciplines

• The way we explore them depends on our background

• A delayed feedback can be used to stabilize a system
8. Delayed negative feedback (Leon)

1. Negative feedback
\[ \frac{dy}{dt} = \frac{1}{1 + y^p} - by \]

A protein represses the transcription of its own gene [for example, PER in the circadian control system of fruit flies]

2. Delayed negative feedback
\[ \frac{dy}{dt} = \frac{1}{1 + y^p (t - \tau)} - by \]

\( \tau \) is the time delay that is required for transcription and translation
Delayed negative feedback

\[
\frac{dy}{dt} = \frac{1}{1 + y^p(t-\tau)} - by
\]

Numerous applications in biology
• Mackey 1996: Periodic crashes in circulating red blood cells (RBC).
  \( p = 7.6 \quad \tau = 1.8 \)
• Pupil eye reflex

Other sciences
El-Nino/Southern-Oscillations (ENSO) variability
Forced DDE for surface Temperature \( T \) (Ghil 2008, 2009)

\[
\frac{dT}{dt} = - \tanh(\kappa T(t-\tau)) + \cos(2\pi t)
\]

Delayed negative feedback atmosphere – ocean coupling
Seasonal cycle in the trade winds
\( \kappa = 100, \tau = 0.01 - 1 \)
\[
\frac{dy}{dt} = \frac{1}{1 + y^p(t - \tau)} - by
\]

1. Steady state:
\[
\frac{1}{1 + y_s^p} - by_s = 0 \quad \rightarrow \quad b = \frac{1}{y_s (1 + y_s^p)}
\]

2. Stability:
\[
y = y_s + u \quad (|u| << 1)
\]
\[
\frac{du}{dt} = -\frac{py_s^{p-1}}{(1 + y_s^p)^2} u(t - \tau) - bu
\]

3. Try: \( u = c \exp(\lambda t) \)
\[
\lambda c \exp(\lambda t) = -\frac{py_s^{p-1}}{(1 + y_s^p)^2} c \exp(\lambda(t - \tau)) - bc \exp(\lambda t)
\]
\[
\lambda = -\frac{py_s^{p-1}}{(1 + y_s^p)^2} \exp(-\lambda \tau) - b
\]
\[
\rightarrow \lambda = -pby_s^{p+1} \exp(-\lambda \tau) - b
\]
is the characteristic equation for \( \lambda \)
\[ b = \frac{1}{y_s(1 + y_s^p)} \]
\[ \lambda = -p b y_s^{p+1} \exp(-\lambda \tau) - b \]

Hopf conditions \( \lambda = i \omega \)
\[ i \omega = -p b y_s^{p+1} \exp(-i \omega \tau) - b \]
Re: \( 0 = -p b y_s^{p+1} \cos(\omega \tau) - b \)
Im: \( \omega = p b y_s^{p+1} \sin(\omega \tau) \)
3 equations for \( b, y_s \) and \( \omega \)
\( p \to \infty \)
\[ b_{1H} \approx \frac{\pi}{2 p \tau} \]
\[ b_{2H} \approx \frac{1}{1 - p^{-1} \ln(p) + O(p^{-1})} \]

\[ p=20 \]
\[
\frac{dy}{dt} = \frac{1}{1 + y^p(t - \tau)} - by
\]

\(p \rightarrow \infty\)

\[
\frac{dy}{dt} = -by + \begin{cases} 
0 & y(t - \tau) > 1 \\
1 & y(t - \tau) < 1 
\end{cases}
\]

\(y_{\min} = \exp(-b\tau)\)

\(y_{\max} = (1 - b^{-1})\exp(-b\tau) + b^{-1}\)
Bifurcation diagrams

\[ y_{\text{min}} = \exp(-b \tau) \]
\[ y_{\text{max}} = (1 - b^{-1})\exp(-b \tau) + b^{-1} \]
\[ p = 20 \]
\[ y_{\text{min}} = \exp(-b\tau) \]
\[ y_{\text{max}} = (1 - b^{-1})\exp(-b\tau) + b^{-1} \]
\[ \text{period} = -b^{-1} \ln \left( \frac{by_\text{max} - 1}{by_\text{min} - 1} \frac{y_\text{min}}{y_\text{max}} \right) \]
9. Square-waves (Laurent)

$2\tau$ – periodic square - waves of scalar DDEs

$$\varepsilon \frac{dx}{dt} = -x + f(x(t-1))$$

$$\varepsilon \equiv \tau^{-1}$$

$$\varepsilon \rightarrow 0$$

$$-x + f(x(t-1)) = 0$$

$$x_{n+1} = f(x_n)$$

Experiments using an optoelectronic oscillator modeled by a scalar DDE
Larger et al. JOSA B 18, 1063 (2001)

\[ \varepsilon x' = -x + f(x(t-1)) \]

\[ f = \beta \left[ 1 + \frac{1}{2} \cos(x(t-1)) \right] \]
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\[ x_{n+1} = f(x_n) \]

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<td>5.30</td>
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Experiments in Besançon
Opto electronic oscillator

\( \tau \)- periodic solution
\[ y' = x \]
\[ \varepsilon x' = -x - \delta y + f(x(t-1)) \]
\[ \varepsilon = 10^{-3} \quad \delta = 8 \times 10^{-3} \]
\[ y' = x \]
\[ \varepsilon x' = -x - \delta y + f(x(t - 1)) \]
\[ \varepsilon = 10^{-3} \quad \delta = 8 \times 10^{-3} \]

Numerical solution

Asymptotic analysis
10. Control theory (Jean-Pierre)

\[ x' = ax \]
\[ x = 0 \text{ stable if } a < 0 \]

Can we do better?
\[ x' = ax + bx(t - 1) \]
control
\[ x' = ax + bx(t - 1) \]

The characteristic equation is
\[ \sigma - a - b\exp(-\sigma) = 0 \]
\[ \sigma \text{ real: } b = \exp(\sigma)(\sigma - a) \]
\[ \sigma \text{ complex: } \sigma = \gamma + i\omega \]
\[ \gamma - a - b \exp(-\gamma) \cos(\omega) = 0 \]
\[ \omega + b \exp(-\gamma) \sin(\omega) = 0 \]
\[ x' = ax + bx(t - 1) \]

The characteristic equation is
\[ \sigma - a - b \exp(-\sigma) = 0 \]
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\[ \gamma - a - b \exp(-\gamma) \cos(\omega) = 0 \]
\[ \omega + b \exp(-\gamma) \sin(\omega) = 0 \]

\[ \frac{\omega}{\gamma - a} = -\tan(\omega) \rightarrow \gamma = a - \frac{\omega}{\tan(\omega)} \]

\[ b = -e^{\exp(\gamma)} \frac{\omega}{\sin(\omega)} \]
\( \sigma = 0: b = -a \)
\( \gamma = 0: a = \frac{\omega}{\tan(\omega)}, \quad b = -\frac{\omega}{\sin(\omega)} \)

Stable if \( a < 1 \)
Act and wait

\[ x' = ax + G(t)x(t-1) \]

\[ G(t) = \begin{cases} 
0 & (0 < t < 1) \\
\frac{t}{b} & (1 < t < 2) 
\end{cases} \]

\[ G(t) = G(t + 2k) \quad k = 1, 2, \ldots \]
Act and wait
\[ x' = ax + G(t)x(t-1) \]

\[ G(t) = \begin{cases} 0 & (0 < t < 1) \\ b & (1 < t < 2) \end{cases} \]

\[ G(t) = G(t + 2k) \quad k = 1, 2, \ldots \]

0 < t < 1: \quad x' = ax, \quad x(0) = x_0 \rightarrow x = x_0 \exp(at) \]

1 < t < 2: \quad x' = ax + bx_0 \exp(a(t-1)), \quad x(1) = x_0 \exp(a) \]

At \( t = 2 \)
\[ x(2) = x_0 \exp(2a)(1 + b \exp(-a)) = \mu x_0 \]
stable if \( |\mu| < 1 \), when \( t = 2k \) increases
The stability domain is limited by the lines \( \mu = \pm 1 \)
Full delayed feedback control
\[ x' = ax + bx(t - 1) \]
stable if \( a < 1 \)

Act and wait control
\[ x' = ax + G(t)x(t - 1) \]
\[ G(t) = \begin{cases} 0 & (0 < t < 1) \\ b & (1 < t < 2) \end{cases} \]
\[ G(t) = G(t + 2k) \quad k = 1, 2, \ldots \]
stable if \( a > 1 \)
Summary

Analytical tools
1. Delayed negative feedback
   • Linear stability and Hopf bifurcation
   • Delay is moderate but feedback can be strong (limit of strong feedback)
2. Square-waves
   • The large delay limit and maps
3. Control theory
   • New forms of delayed feedback are designed